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The Lorentz-Covariant Approximation Method in General Relativity. - I.

R. P. KERR (*)
Trinity College - Cambridge

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Summary. — The Lorentz-covariant approximation method for the field outside a set of localized particles has been analysed. It is found that as well as the usual equations of motion and energy derived by Eistein, Infeld and Hoffman for the quasi-static approximation, there are three further equations, the equations of spin, which must be satisfied by the structural parameters of each particle. These equations also appear as surface integral conditions in the quasi-static approximation. Furthermore, it is shown that it is not necessary to expand the mass, dipole, or spin parameters. those introduced into the lowest approximation being the physical particle parameters. It is only the differential equations satisfied by these that change in the higher orders.

1. – Introduction.

In this paper we shall analyse the covariant approximation method for calculating the equations of motion of localized particles in general relativity. By covariant we mean that the approximation equations are Lorents covariant, not covariant under the full co-ordinate transformation group. Previously, it has been usual to consider the derivatives with respect to the non-covariant time coordinate to be small compared to the spatial derivatives, giving

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the usual quasi-static approximation. EINSTEIN, INFELD and HOFFMAN ([1-3]) analysed this problem and showed that each particle must satisfy three equations of motion and an equation of mass. Their method was based on the dipole procedure wherein a dipole field is introduced into each approximation, so that the field equations are integrable in the next approximation. However, their proof assumed the vanishing of the dipole forces and, as has been shown by MOFFAT and KERR ([4]), this is not so. In the lowest approximation one obtains the usual classical dipole forces on a non-symmetric particle in an external field.

Furthermore, as I shall show in a subsequent paper, the four surface integral conditions of Einstein, Infeld and Hoffman are not sufficient for the integrability of the approximation equations. There are also three equations, corresponding to the classical equations of angular momentum, which must be satisfied in every approximation. If a dipole field is introduced into each approximation so that the E.I.H. surface integrals are zero, then the three spin equations will be inconsistent. It is also necessary to introduce a spin field into each approximation, before the field equations can be satisfied. Physically, it is found that a dipole particle will start to rotate under an applied couple.

We shall not consider the quasi-static approximation further in this paper but instead shall analyse the corresponding problem in the Lorentz-covariant approximation. The reason for this is that, although it is much more difficult to integrate the field equations in a simple closed form, we shall have a clearer picture of the structure of the approximation method. In particular, we shall see that the equations of energy, motion, and spin are the only physical equations to be satisfied by the particle constants. The form of these equations is completely identical to that in the quasi-static approximation, except that the time derivative is treated in the same way as the spatial derivatives.

In Sections **2** and **3** we shall show that the field equations may be integrated in each approximation, provided that we expand the mass, dipole and spin parameters for each particle. It will be shown that each of these parameters must satisfy a first or second order differential equation, as in the quasi-static approximation. However, it can be shown that the total field is a function of the mass, etc., rather than of the individual n-th approximation order components. This being so, we would expect that the physical equations of mass, motion, and spin would also be functions of these total mass, dipole and spin constants, and that the differential equations satisfied by the n-th order para-

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([1]) A. EINSTEIN, L. INFELD and *H.* HOFFMAN: *Ann. Math.*, **39**, 66 (1938).
([2]) A. EINSTEIN and L. INFELD: *Ann. Math.*, **41**, 797 (1940).
([3]) A. EINSTEIN and L. INFELD: *Can. Journ. Math.*, **1**, 109 (1949).
([4]) J. MOFFAT and R. KERR: paper. Copies available.

meters do not affect the physical field. This was the motivating idea behind the E.I.H. dipole procedure. However, they attempted to show that only the total dipole constants would appear in the equations of motion by proving that the dipole forces were zero in every approximation, which is not true as has been shown in [4].

As we shall see in Sect. 4, exactly the same problem arises in the Lorentz-covariant approximation. It could be proved that it is only the total particle parameters that appear in the physical equations but we shall not do so. Instead, we notice that if the individual n-th order parameters do not appear in the physical field or the equations of motion, etc., it is pointless to expand these parameters. Of course, if we do not expand them then we cannot satisfy the approximation equations exactly. However, we show that if the equations of motion, energy and spin are satisfied to the n-th approximation then the Einstein field equations are also satisfied *to the same approximation.* The original n-th approximation field equations of E.I.H. are not satisfied even in an approximate sense but, if we add together the n-th approximation fields up to the N-th order to give the physical field to the N-th approximation, then the field equations of Einstein will be satisfied to the N-th approximation, provided that the seven differential equations of energy, motion and spin are satisfied *to the same approximation.* This is all that can be expected of any approximation method.

2. – Lorentz-invariant λ-expansions.

In this Section we shall derive an approximation method for solving the field equations outside a set of weakly interacting, fast moving, bodies. Obviously, if we wished to consider fundamental particles we should have to take into account the Bremstrahlung emission which plays a large part in the theory. This is also the case with classical theories, such as quantum theory or electrodynamics, where the spontaneous emission of radiation plays an important role in the motion of the elementary particles.

For a macroscopic body, the rate of emission will be governed by the equations of statistical mechanics, and consequently may be considered to be a continuous function of the time for a stable body in a weak gravitational field. Thus, it should be possible to derive meaningful equations of motion for such bodies.

In the following, we shall consider the field around a set of particles, or « singularities », which we shall distinguish by Latin superfixes over the appropriate functions. As it is not possible to expand the field at the singularities, and as we do not wish to introduce an energy-momentum tensor, we shall enclose the particles by arbitrary three dimensional surfaces with the topo-

logical properties of a time-like cylinder. To specify the position of the p-th particle, we shall take a representation world line, $\overset{p}{z}_\mu(\overset{p}{s})$, inside the p-th surface, $\overset{p}{S}$. Since the actual shape of the surfaces is not important, we shall define them by the following conditions.

Firstly, we define a one dimensional parameter, $\overset{p}{s}$, along each world line by means of the Galilean metric tensor (*). We then extend the domain of definition of this parameter to a four dimensional region, enclosing the particle, by means of the equations,

$$x^\mu = \overset{p}{z}_\mu(\overset{p}{s}) + \gamma^\mu \,, \tag{2.1}$$

$$\overset{p}{v}_\mu \gamma^\mu = \overset{p}{v}_\mu (x^\mu - \overset{p}{z}_\mu(\overset{p}{s})) = 0 \,, \tag{2.2}$$

where throughout this paper we shall raise and lower the particle tensors with the Galilean tensor, rather than the unknown Einstein tensor. These equations correspond to taking the planes of constant $\overset{p}{s}$ perpendicular to the world line, and they will have a unique solution for $\overset{p}{s}$ as a function of x^α inside S provided that the derivative, $\partial s/\partial x^\mu$, is not infinite. If we differentiate (2.2) with respect to the parameter x^α, we have the equation

$$\frac{\partial s}{\partial x^\alpha} \dot{v}_\beta \gamma^\beta + v_\beta \left(\delta^\beta_\alpha - v^\beta \cdot \frac{\partial s}{\partial x^\beta} \right) = 0 \,, \tag{2.3}$$

and so

$$\frac{\partial s}{\partial x^\alpha} = \frac{v_\alpha}{1 - (\dot{v}_\nu \gamma^\nu)} \,, \tag{2.4}$$

so that s will be a single valued function of position provided that $\dot{v}_\nu \cdot \gamma^\nu < 1$ inside the surface. We define the surface, $\overset{p}{S}$, by the condition that $(\gamma_\alpha \cdot \gamma^\alpha)$ is a constant everywhere on the surface; *i.e.* in the rest system of the world line the surface has constant Galilean radius. The condition for single valuedness means that the acceleration of the world line should be small compared to the dimensions of the system. We do not want to take a world line oscillating rapidly across the particle, as we should then have to disentangle the violent motion of this form the slow acceleration of the physical body. This means that we must find some way of fixing the position of the world line inside the particle. We shall leave this to a later section.

We shall impose the De Donder co-ordinate conditions on the fundamentatensor

$$\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0 \,, \tag{2.5}$$

(*) $\eta_{\alpha\beta}$ is the Galilean metric tensor, $(+1, -1, -1, -1)$; $(\mathrm{d}s)^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$, and $v^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$. We shall denote differentiation with respect to s by a dot, e.g. $\dot{v}^\alpha = \mathrm{d}v^\alpha/\mathrm{d}s$. Where there is no ambiguity, we shall omit the particle index, p.

where

$$\mathfrak{g}^{\mu\nu} = (-g)^{\frac{1}{2}} \cdot g^{\mu\nu},$$

and $g = \det(g_{\alpha\beta})$. Later, we shall consider whether this introduces any non-physical equations into the theory. We are not concerned here with whether these conditions have any physical significance of their own, since, whether this is so or not, they are the most convenient to use in any reiteration procedure. From the explicit expression for the curvature tensor in Appendix A, we see that the approximation field equations reduce to D'Alembert equations whenever these co-ordinate conditions are satisfied.

It is fundamental in the solution of any non-linear equation by successive approximation that the field equations should first be expressed in terms of a set of independent variables. We shall take

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \tag{2.6}$$

and express the field equations in terms of the $h^{\alpha\beta}$. FOCK has shown that, in a co-ordinate system satisfying (2.5), the field equations reduce to

$$\Box h^{\mu\nu} = 2\Lambda^{\mu\nu}, \tag{2.7}$$

where $\Lambda^{\mu\nu}$ is a non-linear function of the $h^{\alpha\beta}$ and their derivatives. The complete expression for the Einstein tensor, as well as its reduced form using (2.5), is given in Appendix A. It may be shown that the $\Lambda^{\mu\nu}$ are absolutely convergent power series in the $h^{\alpha\beta}$, provided that the moduli of the characteristic roots of $\eta_{\mu\alpha} h^{\alpha\nu}$ are all less than unity. A simpler, though not so sensitive, condition is that

$$\mod(h^{\alpha\beta}) < \tfrac{1}{4}. \tag{2.8}$$

The « obvious » way to solve (2.7) is to use *successive reiteration*,

$$\Box \underset{(n)}{h^{\mu\nu}} = 2 \overset{*}{\underset{(n)}{\Lambda}}{}^{\mu\nu}(\underset{(n-1)}{h^{\alpha\beta}}), \tag{2.9}$$

where $\overset{*}{\underset{(n)}{\Lambda}}{}^{\mu\nu}$ consists of all terms in the expansion of the field equations which do not contain more than $n(h^{\alpha\beta})$. However, there are two things wrong with this intuitive approach. The first is that there would be too much unnecessary labour involved if we intended to stop at the N-th reiteration, since it would be sufficiently accurate to use $(\underset{(1)}{h^{\cdot\cdot}})^N$ in the last reiteration rather than $(\underset{(N-1)}{h^{\cdot\cdot}})^N$. The difference is negligible in the N-th reiteration. This would not matter if we intended to carry the process on to infinity (whatever that may

mean in a logical sense!), but in practise one has to cut the process off at some finite reiteration.

The second thing wrong with this approach is that we have to insure that the limiting field, $\underset{(\infty)}{h}^{\mu\nu}$, will satisfy the co-ordinate conditions. On the surface, the simplest way to do this seems to be to make each successive approximation satisfy it exactly, *i.e.*

$$\underset{(n)}{h}^{\mu\nu}{}_{,\nu} = 0 \,, \tag{2.10}$$

but, as we shall see in the next section, this is not so. Nevertheless, for the present we shall assume that (2.10) is to be satisfied. If we take the divergence of (2.9) we see that $\overset{*}{\underset{(n)}{\Lambda}}{}^{\mu\nu}{}_{,\nu}$ must be zero, and this will not be so, in general, even if the co-ordinate conditions are satisfied exactly in every previous reiteration.

Fortunately, the classical solution of the first problem is also the solution to the second. In the standard work on non-linear equations by Bogoliubov, we see that the usual method used to solve such differential equations is to expand the field in powers of an *indeterminate parameter*, λ,

$$h_{\mu\nu}(\lambda) = \sum_r \lambda^r \underset{r}{h}^{\mu\nu} \,, \tag{2.11}$$

which corresponds to the expansion method introduced by EIHSTEIN, INFELD and HOFFMAN in 1938 for the quasi-static approximation. It must be stressed that this « parameter » has no physical significance whatsoever. The λ-series in (2.11) is not the physical field — that is obtained by replacing λ by unity. The significance of λ is that it should act as a weighting factor to define the order of magnitude of different terms in the approximation procedure, thereby reducing the labour involved in the reiteration methods. Some times, when there is an obvious constant present in the differential equations, it is convenient to expand in terms of this rather than an indeterminate constant, the physical constant playing the role of a weighting factor. However, there is no obvious parameter to chose in the covariant approximation.

If we expand the field in powers of this parameter, and then equate to zero the coefficient of λ^n in the field equations, we obtain a new set of approximation equations,

$$\Box \underset{n}{h}^{\mu\nu} = 2 \underset{n}{\Lambda}^{\mu\nu} \,. \tag{2.12}$$

Hence, the *reiteration* field must satisfy the equation,

$$\tfrac{1}{2} \Box \underset{(n)}{h}^{\mu\nu} = \underset{(n)}{\Lambda}^{\mu\nu}(\underset{(r)}{h}^{\alpha\beta};\, r < n) = \sum_r^n \underset{r}{\Lambda}^{\mu\nu} \,, \tag{2.13}$$

where

$$\underset{(n)}{h}^{\mu\nu} = \sum_{1}^{n} \underset{r}{h}^{\mu\nu}, \tag{2.14}$$

and $\underset{n}{\Lambda}^{\mu\nu}$, and so $\underset{(n)}{\Lambda}^{\mu\nu}$, will be a finite polynomial in the previous approximations.

We shall now state an important theorem that was first proved by EINSTEIN and INFELD, in 1949, for the quasi-static approximation.

> *Theorem 1.* If the gravitational field equations of Fock are expanded as a power series in an indeterminate parameter, and the coefficients of the successive powers equated to zero, (2.12), then the divergences, $\underset{n}{\Lambda}^{\mu\nu}{}_{,\nu}$ and $\underset{(n)}{\Lambda}^{\mu\nu}{}_{,\nu}$, will both be zero, provided that the field equations and the co-ordinate conditions are satisfied in all previous approximations.

This follows from the Bianchi identities and is proved in a more general form in Sect. **4**. It follows immediately that, provided that (2.12) and (2.10) are satisfied in every previous approximation, any solution of the field equations in the n-th approximation must satisfy

$$\Box \underset{,\nu}{h}^{\mu\nu} = 0, \tag{2.15}$$

as may be proved by taking the divergence of (2.12). This does not imply that the co-ordinate conditions are satisfied by any solution of the field equations. In the next section, however, we shall see that the De Donder conditions may be satisfied exactly in the n-th approximation by the addition of a solution of the homogeneous wave equation onto any particular solution of (2.16). This will give the equations of mass, motion, and spin.

3. – The solution of the field equations.

Firstly, we must find a particular solution of the field equations outside the surfaces, $\overset{2}{S}$. The final results will, however, be independant of the surface chosen, as we can always use analytic continuation to extend the approximation field inside the surface. Of course, the solution obtained in this manner would be singular at the world lines, and so physically meaningless there. Even for the Schwarzschild metric, when we take a finite number of terms in this asymptotic expansion, the successive approximation fields bear very little relationship with the exact solution at and near the singularity.

Let $D(x^\varkappa)$ be an invariant Green's function satisfying the wave equation everywhere except the origin. We shall not specify it at present, as the choice is governed by the boundary conditions at infinity or by causal considerations.

We shall write our particular solution of (2.12) in the form,

$$\overset{*}{h}{}^{\mu\nu}_{n} = (2\pi)^{-1} \cdot \int_{V} \Delta^{\mu\nu}_{n} \cdot D(r - r')\, \mathrm{d}^4 r' \,, \tag{3.1}$$

where V is the 4-volume outside the surfaces. To obtain the general solution of the field equations, we must add an arbitrary solution of the wave equation onto $\overset{*}{h}{}^{\mu\nu}_{n}$,

$$h^{\mu\nu}_{n} = \overset{*}{h}{}^{\mu\nu}_{n} + U^{\mu\nu}_{n} \,, \tag{3.2}$$

where,

$$\Box U^{\mu\nu}_{n} = 0 \,. \tag{3.3}$$

From (3.1) we see that

$$\overset{*}{h}{}^{\mu\nu}_{n\,,\nu} = (2\pi)^{-1} \int_{V} \Delta^{\mu\nu}_{n}(r') \cdot \frac{\partial}{\partial x^{\nu}} \cdot D(r - r') \cdot \mathrm{d}^4 r' \,. \tag{3.4}$$

Since the Green's function is a function of $(r - r')^2$, we have

$$\frac{\partial}{\partial x^{\nu}} \cdot D(r - r') = -\frac{\partial}{\partial x'^{\nu}} D(r - r') = D(r - r')_{\,,\nu} \,, \tag{3.5}$$

by definition, and so we may reduce (3,3), by integration by parts, to

$$\begin{aligned} \overset{*}{h}{}^{\mu\nu}_{n\,,\nu} &= -(2\pi)^{-1} \int_{V} \Delta^{\mu\nu}_{n}(r') \cdot \frac{\partial}{\partial x'^{\nu}} \cdot D(r - r') \cdot \mathrm{d}^4 r' = \\ &= (2\pi)^{-1} \int_{V} \Delta^{\mu\nu}_{n\,,\nu}(r') \cdot D(r - r') \cdot \mathrm{d}^4 r' + (2\pi)^{-1} \sum_{p} \int_{p} \Delta^{\mu\nu}_{n} \cdot D \cdot \mathrm{d}S'_{\nu} \,, \end{aligned} \tag{3.6}$$

where the p on the integral sign denotes the surface S^p, and $\mathrm{d}S'_{\nu}$ is an element of the surface S^p at the point r', pointing along the same direction as the vector γ_{ν}. Therefore, provided that the field equations and co-ordinate conditions are satisfied in all previous approximations,

$$h^{\mu\nu}_{n\,,\nu} = (2\pi)^{-1} \sum_{p} \int_{p} \Delta^{\mu\nu}_{n}(r') \cdot D(r - r') \cdot \mathrm{d}S'_{\nu} + U^{\mu\nu}_{n\,,\nu} \,. \tag{3.7}$$

We shall use (2.1-3) to reduce this surface integral to a line integral along the world lines. We can expand $D(r - r')$, at the point (s, γ^{α}), as a Taylor's series,

$$D(r - r') = \sum_{t=0}^{\infty} \frac{(-)^t}{t!} \cdot D\big(r - z(s)\big)_{,(\alpha)_t} \cdot \gamma^{(\alpha)_t} \,, \tag{3.8}$$

where we have used the following notation,

$$F(x^\varrho)_{,(\alpha)_t}\cdot\gamma^{(\alpha)_t} = F(x^\varrho)_{,\alpha_1\alpha_2\ldots\alpha_t}\cdot\gamma^{\alpha_1}\cdot\gamma^{\alpha_1}\ldots\gamma^{\alpha_t}\,. \tag{3.9}$$

Since the surfaces have constant radius and since each S^p is orthogonal to the planes of constant $\overset{p}{s}$, we may take as the surface element $\mathrm{d}^2S\cdot\mathrm{d}u\cdot n_\nu$. d^2S is a two dimensional element on the surface, $(\gamma)^2 = \text{const.}$, $s(r) = \text{const.}$; $\mathrm{d}u$ is the distance between neighbouring planes, measured along the direction v^α at the point r'; n^α is a unit four vector with direction γ^α. This may be seen by considering the surface element in the rest system of the p-th particle. From (2.3), we see that

$$\mathrm{d}s = \frac{v_\alpha(v^\alpha\cdot\mathrm{d}u)}{1-\dot{v}_\alpha\cdot\gamma^\alpha}\,,$$

and therefore

$$\mathrm{d}u = (1-(\dot{v},\gamma))\,\mathrm{d}s\,, \tag{3.10}$$

where we use the notation,

$$(\dot{v},\gamma) = \dot{v}_\alpha\cdot\gamma^\alpha\,.$$

Consequently, the surface integral over the p-th singularity will reduce to

$$\overset{*}{\underset{n}{h}}{}^{\mu\nu}{}_{,\nu} = (2\pi)^{-1}\sum_{p,t}\frac{(-)^t}{t!}\int\overset{p}{\underset{n}{A}}{}^{\mu\nu}\cdot n_\nu\cdot D(r-r')_{,(\alpha)_t}\gamma^{(\alpha)_t}\cdot(1-(\dot{v},\gamma))\,\mathrm{d}^2S\cdot\mathrm{d}s = \tag{3.11}$$

$$= \sum_{p,t}\int\overset{p}{\underset{n}{k}}{}^{\mu;(\alpha)_t}\cdot D(r-z)_{,(\alpha)_t}\cdot\mathrm{d}s\,,$$

where the k's are defined by the equations,

$$\overset{p}{\underset{n}{k}}{}^{\mu;(\alpha)_t} = \frac{(-)^t}{2\pi t!}\int\overset{p}{\underset{n}{A}}{}^{\mu\nu}(s,\gamma)\cdot n_\nu\cdot\gamma^{(\alpha)_t}\cdot(1-(\dot{v},\gamma))\cdot\mathrm{d}^2S\,, \tag{3.12}$$

and are completely symmetric in their indices, $(\alpha)_t$.

So far we have not specified the complementary function, $\underset{n}{U}{}^{\mu\nu}$. This may be either singular at the world lines or regular throughout space. Before we can eliminate this regular function from the field, it is necessary to define the boundary conditions at infinity. The natural hypothesis to make is that either there is no radiation at infinity, or there is no incoming radiation. Provided that one of these assumptions is made we can eliminate the arbitrary non singular wave function by choosing the appropriate Green's function. Under these circumstances, the complementary function for the field equations, (2.12), may be reduced to a line integral. To see this, we notice that, by the appropriate Green's theorem, the complementary function may be written as the integral of a function, linear in the $D(r-r')$ and its normal derivative, over the surfaces, S^p. By the methods of the previous paragraphs, this may

be reduced to a line integral along the world lines in the same way as (3.6) was reduced to (3.11).

If the physical system is such that there is radiation coming in from infinity, it can be introduced in the first approximation by the addition of an arbitrary non-singular function — it will not affect the following arguments. Consequently, we shall only look for a solution of (2.12) such that the co-ordinate conditions are satisfied, there being no point in looking for the most general solution in every approximation. Any arbitrariness in the n-th approximation is already present in the first approximation, so that the physical character of the solution is determined by $\underset{1}{h}^{\mu\nu}$.

Because of these considerations, we shall write the complementary function as a line integral,

$$\underset{n}{U}^{\mu\nu} = \sum_{t,p} \int \underset{n}{M}^{\mu\nu;(\alpha)_t} \cdot D(r-z)_{,(\alpha)_t} \cdot \mathrm{d}s \,, \tag{3.13}$$

where the M's are completely symmetric in the α's, as well as in μ and ν. If we take the divergence of (3.13), we have

$$\underset{n}{U}^{\mu\nu}{}_{,\varrho} = \sum_{t,p} \int \underset{n}{M}^{\mu\nu;(\alpha)_t} \cdot D(r-z)_{,\varrho,(\alpha)_t} \cdot \mathrm{d}s \,, \tag{3.14}$$

so that the co-ordinate conditions may be satisfied, provided that

$$\sum_{t,p} \int (\underset{n}{M}^{\mu\nu;(\alpha)_t} + \underset{n}{k}^{\mu;\nu(\alpha)_t}) \cdot D(r-z)_{,\nu,(\alpha)_t} \cdot \mathrm{d}s + \sum_{p} \int \underset{n}{k}^{\mu} \cdot D(r-z) \cdot \mathrm{d}s = 0 \,, \tag{3.15}$$

where the particle tensors, $M^{\cdot\cdot}$, may be chosen as arbitrary functions of the world lines parameter, s. We shall chose particular solutions for these parameters so that this equation is satisfied. When this is done, we shall have the complete solution of the field equations and co-ordinate conditions in the n-th approximation if we add any solution of the wave equation which also satisfies the co-ordinate conditions, onto this expression.

First, we observe that

$$\begin{aligned} \int f(s) \cdot v^{\varrho} \cdot F(r-z)_{,\varrho} \cdot \mathrm{d}s &= -\int f(s) \cdot v^{\varrho} \cdot \frac{\partial}{\partial z^{\varrho}} \cdot F(r-z) \cdot \mathrm{d}s = \\ &= -\int \frac{\mathrm{d}}{\mathrm{d}s} F(r-z) \cdot f(s) \cdot \mathrm{d}s = \int \dot{f} \cdot F(r-z) \cdot \mathrm{d}s \,, \end{aligned} \tag{3.16}$$

where $F(r-z)$ is any function that vanishes sufficiently rapidly in a time

like direction. We define $\overset{p}{W}{}^{\mu\nu}_{n}$ by the equation,

$$(3.17)(*) \qquad W^{\mu\nu} = \int^{p} (A^{\varrho} v^{\mu} v^{\nu} - A^{\mu} v^{\varrho} v^{\nu} - A^{\nu} v^{\varrho} v^{\mu}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,.$$

It is symmetric in μ and ν, and

$$(3.18) \qquad W^{\mu\nu}{}_{,\nu} = -\int (A^{\mu} v^{\nu} v^{\varrho}) D(r - z)_{,\varrho\nu} \cdot \mathrm{d}s = \\ = -\int \dot{A}^{\mu} \cdot D(r - z) \cdot \mathrm{d}s - \int A^{\mu} \cdot v \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,,$$

so that, if we define A^{μ} by the equation,

$$(3.19) \qquad \frac{\mathrm{d}^2}{\mathrm{d}s^2} (A^{\mu}) = k^{\mu} \,,$$

we can eliminate the monopole term in (3.15) by adding $W^{\mu\nu}$ onto the original solution. The term in $D_{,\nu}$ will introduce a further complication into the theory, since, if we equate its coefficient to zero in (3.15),

$$(3.20) \qquad M^{\mu\nu} + k^{\mu;\nu} - A^{\mu} \cdot \dot{v}^{\nu} = 0 \,,$$

we cannot satisfy this equation with a *symmetric* $M^{\mu\nu}$. However, if we write

$$(3.21) \qquad Q^{\mu\nu} = \int (F^{\mu\varrho} \cdot v_{\nu} \, \mathrm{c} \, F^{\nu\varrho} \cdot v^{\mu}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,,$$

where $F^{\mu\nu}$ is antisymmetric in μ and ν, we see that

$$(3.22) \qquad Q^{\mu\nu}{}_{,\nu} = \int \frac{\mathrm{d}}{\mathrm{d}s} (F^{\mu\varrho}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,,$$

This shows that we can eliminate the antisymmetric part of (3.20) by adding $Q_{\mu\nu}$ onto the original solution, provided that

$$(3.23)(**) \qquad \frac{\mathrm{d}}{\mathrm{d}s} (F^{\mu\nu}) = -k^{[\mu;\nu]} + A^{[\mu} \cdot \dot{v}^{\nu]} \,,$$

(*) For the sake of clarity, we shall omit the particle index, p, and the order index, n, for the rest of this section.

(**) We shall use the standard notation, $2.A^{[\mu}B^{\nu]} = A^{\mu}B^{\nu} - A^{\nu}B^{\mu}$, and $2.A^{(\mu}B^{\nu)} = A^{\mu}B^{\nu} + A^{\nu}B^{\mu}$.

and we can eliminate the symmetric part of (3.20) by choosing an $M^{\mu\nu}$ such that

$$M^{\mu\nu} = -k^{(\mu:\nu)} + A^{(\mu}\cdot\dot{v}^{\nu)}. \tag{3.24}$$

The higher order multipoles in (3.15) offer no problems whatsoever as they may be eliminated by adding to the field

$$\sum_i \int (k^{\varrho:\mu\nu(\alpha)_i} - k^{\mu:\varrho\nu(\alpha)_i} - k^{\nu:\varrho\mu(\alpha)_i}) D(r-z)_{,\varrho(\alpha)_i}\cdot ds\,, \tag{3.25}$$

which is symmetric in μ and ν, because the k's are symmetric in all their indices after the colon. Finally, we obtain a solution of the equations in the n'th approximation by adding onto the original integral expression, (3.1), the additional terms, (3.17), (3.21), (3.24), and (3.25). There are, of course, other singular function that we can add to (3.1) to remove the monopole and dipole terms, but we should still have to solve differential equations like (3.19) and (3.23) to satisfy the coordinate conditions. The reason for this is that we cannot remove the monopole, k^{μ}, in general by adding a monopole term to the original field, and so we have to add a dipole. When we insert this into the co-ordinate conditions we obtain a quadrupole expression, and the only way that this can be reduced to a monopole is through the application of equation (3.16) twice. Consequently, we obtain the second derivatives of the dipole constants. The reason that this reduction of a quadrupole to a monopole is possible is that the derivatives of the Green's function are not linearly independant under the integral sign but are related by (3.16) as well as $\eta^{\alpha\beta}\cdot D(r-z)_{,\alpha\beta} = 0$.

> *Theorem 2.* The field equations and co-ordinate conditions may be satisfied in the n'th approximation, provided that they are satisfied in every previous approximation. In general, however, there will be a second order differential equation to satisfy, and so the solution will be finite in space like directions only.

The first part of the theorem has been proved, as we have found a particular solution of the field equations which satisfies the co-ordinate conditions. To prove the second part we observe that A^{μ} must satisfy a second order differential equation and so it will diverge at infinity on the world line. This is the case in the quasi-static approximation where the second order dipole moment may be interpreted as the distance between the physical world line and the world line obtained by integrating the equations of motion in the fourth approximation. The solution will be finite only if the system is periodic.

The answer to this is that the individual $\underset{n}{A}$'s and $\underset{n}{F}$'s are not physical

tensors, but have only been introduced so that the coordinate conditions may be satisfied *exactly* in each approximation. This is a rather naive situation as the field equations have only been satisfied approximately, *i.e.* to the n'th approximation. What is wanted is an approximation method in which the coordinate conditions are satisfied to the same approximation as the field equations, but no further. We shall derive a reiteration method in the next chapter which does this without it being necessary to introduce the non-physical $\underset{n}{A}$'s and $\underset{n}{F}$'s. It must first be stressed, however, that these differential equation do *not* arise because we have chosen the De Donder coordinate conditions. Unless these conditions are satisfied, *exactly*, the n'th approximation equations of Einstein are not satisfied since we have used the coordinate conditions to reduce Einstein's equations to the simpler Fock equations. However, if the co-ordinate conditions are satisfied to the n'th approximation, the solution we derive will satisfy Einstein's gravitational equations to the n'th approximation and that is all we should expect from an approximation method!

4. – The equations of motion.

If we sum equations (3.19) and (3.23) over all approximation orders, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(A^\mu) = \sum_{n=1}^{\infty} \overset{p}{\underset{n}{k}}{}^{\mu}(\overset{q}{\underset{r}{A}}{}^{\alpha}, \overset{q}{\underset{r}{F}}{}^{\alpha\beta};\ r < n) \overset{\text{def.}}{=} \overset{p}{k}{}^{\mu}(\overset{q}{\underset{r}{A}}{}^{\alpha}, \overset{q}{\underset{r}{F}}{}^{\alpha\beta}), \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\overset{p}{F}{}^{\mu\nu}) = \sum_{n=1}^{\infty} (\overset{p}{\underset{n}{k}}{}^{[\nu:\mu]} + \overset{q}{\underset{n}{A}}{}^{[\mu}\overset{p}{\dot{v}}{}^{\nu]}) \overset{\text{def.}}{=} \overset{p}{k}{}^{[\nu:\mu]} + \overset{p}{A}{}^{[\mu}\overset{p}{\dot{v}}{}^{\nu]}, \tag{4.2}$$

where we have defined

$$\overset{p}{A}{}^{\mu} = \sum_{n-1} \overset{p}{\underset{n}{A}}{}^{\mu}, \qquad \overset{p}{F}{}^{\mu\nu} = \sum_{n-1} \overset{p}{\underset{n}{F}}{}^{\mu\nu}. \tag{4.3}$$

What we should like to say is that the individual n'th order constants have no physical meaning, but only the total $\overset{p}{A}{}^{\mu}$ and $\overset{p}{F}{}^{\mu\nu}$, defined in (4.3). Provided that the right hand sides of equations (4.1) and (4.2), as well as the limiting field, $\underset{(\infty)}{h}{}^{\mu\nu}$, are functions of the *total* spin and dipole moments and not of the individual n-th order parameters, we could interpret these equations as the *physical* equations of motion and spin. Under these circumstances, we should not be interested in the solutions of (3.19) and (3.23) at all, but only in the solution of (4.1-2).

Now, it can be proved that it is only the total spin and dipole parameters that appear in the limiting field and the equations of motion. The reason that we have had to introduce the $\underset{n}{A}$'s, etc., is that we have been trying to satisfy the coordinate conditions *exactly* in each approximation. What we shall now prove is that, if we weaken this condition so that the field to the n'th approximation does not satisfy these conditions exactly, but only to the n'th approximation, we shall not need to expand out these parameters at all. It will be found that the co-ordinate conditions will be satisfied to the n'th order provided that *three* equations of spin and four equations of motion and mass are satisfied to the same order.

We shall define the reiteration $\underset{(n)}{k}$'s by the equations,

$$(4.4)\ (^*) \qquad \overset{p}{\underset{(n)}{k}}{}^{\mu}_{\cdot\cdot} = \sum_{r=1}^{n} \overset{p}{\underset{r}{k}}{}^{\mu}_{\cdot\cdot}, \qquad \overset{p}{\underset{(n)}{k}}{}^{\mu\nu} = \sum_{r=1}^{n} \overset{p}{\underset{r}{k}}{}^{[\mu\nu]},$$

where the $\underset{r}{k}$'s will be defined by induction. They correspond to the k's of the previous section but they do not contain the individual $\underset{n}{A}$'s or $\underset{n}{F}$'s. This does not mean that we shall only consider particles whose spin and dipole constants are zero, since the *physical* constants may be introduced into the first approximation, and both the reiteration field and the k's will be functions of these first order constants. What we shall not do is expand these physical tensors into non-physical components of different orders.

We shall now prove, by induction, that it is possible to solve the approximation equations in such a way that $\underset{(n)}{\mathfrak{g}}{}^{\mu\nu}{}_{,\nu}$ is a function in which every term consists of a $\underset{(r)}{k}$ multiplied by a function of the $(n-r)$-th approximation order. When we equate the sum of the k's to zero, for each particle,

$$(4.5) \qquad \overset{p}{\underset{(\infty)}{k}}{}^{\mu}_{\cdot\cdot} = \sum_{n=1}^{\infty} \overset{p}{\underset{n}{k}}{}^{\mu}_{\cdot\cdot} = 0\,, \qquad \overset{p}{\underset{(\infty)}{k}}{}^{\mu\nu} = \sum_{n=1}^{\infty} \overset{p}{\underset{n}{k}}{}^{\mu\nu} = 0\,,$$

we shall have

$$(4.6) \qquad \overset{p}{\underset{(r)}{k}}{}^{\mu}_{\cdot\cdot} = -\sum_{r+1}^{\infty} \overset{p}{\underset{r}{k}}{}^{\mu}_{\cdot\cdot},$$

where every term on the right hand side is of the $(r+1)$-th, or higher, order. Consequently, every term in $\underset{(n)}{\mathfrak{g}}{}^{\mu\nu}{}_{,\nu}$ will be zero *to the n-th order*, and so the co-ordinate conditions will be satisfied to that order. Hence, provided that the

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(*) In the later part of this section, we shall show that $\underset{n}{k}{}^{\mu\nu}v_{\mu} = 0$ and so there are only three independent components.

solution is convergent, the co-ordinate conditions will be satisfied by the limiting field, whenever equations (4.5) are satisfied.

FOCK has shown that the Einstein tensor may be reduced to

$$2\mathfrak{G}^{\mu\nu} = -\Box h^{\mu\nu} + 2\Lambda^{\mu\nu} + Z^{\mu\nu}, \tag{4.7}$$

where

$$Z^{\mu\nu} = h^{\alpha\beta}{}_{,\beta} \cdot P^{\mu\nu}_{\alpha} + h^{\alpha\beta}{}_{,\beta\varrho} \cdot Q^{\mu\nu:\varrho}_{\alpha}. \tag{4.8}$$ (*)

The Bianchi identities may be written as

$$-\Box h^{\mu\nu}{}_{,\nu} + 2\Lambda^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\beta}(-\Box h^{\alpha\beta} + 2\Lambda^{\alpha\beta}) \equiv$$
$$\equiv -h^{\alpha\beta}{}_{,\beta}(P^{\mu\nu}_{\lambda}{}_{,\nu} + \Gamma^{\mu}_{\varrho\nu} P^{\varrho\nu}_{\lambda}) - h^{\lambda\beta}{}_{,\beta\gamma}(P^{\mu\gamma}_{\alpha} + Q^{\mu\nu:\gamma}_{\lambda}{}_{,\nu} + \Gamma^{\mu}_{\varrho\nu} Q^{\varrho\nu:\gamma}_{\lambda}) - h^{\lambda\beta}{}_{,\beta\varrho\nu} Q^{\mu\nu:\varrho}_{\lambda}, \tag{4.9}$$

where the important thing to notice is that the right hand side contains the divergence of the field tensor as a linear factor in each term. If we insert the λ-series, (2.11), into this identity, and then equate the coefficient of λ^n to zero, we have

$$2\underset{n}{\Lambda}{}^{\mu\nu}{}_{,\nu} + \sum_{r>0} \underset{r}{\Gamma}{}^{\mu}_{\alpha\beta}(-\Box \underset{n-r}{h}{}^{\alpha\beta} + 2\underset{n-r}{\Lambda}{}^{\alpha\beta}) = -\sum_{r=1}^{n-1} \underset{n-r}{h}{}^{\alpha\beta}{}_{,\beta}(\underset{r}{P}{}^{\mu\nu}_{\alpha,\nu} + \underset{r}{\Gamma}{}^{\mu}_{\varrho\nu} P^{\varrho\nu}_{\lambda}) - \ldots, \tag{4.10}$$

where there are six terms on the R.H.S., all linear in $\underset{r}{h}{}^{\alpha\beta}{}_{,\beta}$ or its derivatives. It might be wondered why we have omitted the first term in (4.9) from (4.10). The reason is that, since (4.9) is an identity, it must cancel with the terms on the R.H.S. which are linear in the derivatives of $\underset{(n)}{h}{}^{\lambda\beta}$. This is also the reason why we have started the summation in (4.10) from $r = 1$ rather than from $r = 0$.

When we sum these equations from 1 to n, we have

$$2\underset{(n)}{\Lambda}{}^{\mu\nu}{}_{,\nu} + \sum \underset{n-r}{\Gamma}{}^{\mu}_{\alpha\beta}(-\Box \underset{(r)}{h}{}^{\alpha\beta} + 2\underset{(r)}{\Lambda}{}^{\alpha\beta}) \equiv -\sum_{r=1} \underset{(n-r)}{h}{}^{\alpha\beta}{}_{,\beta} \cdot \underset{r}{P}{}^{\mu\nu}_{\alpha,\nu} - \ldots, \tag{4.11}$$

where we must be careful to distinguish between the reiteration and approximation indices, (t) and t respectively.

Now, let us assume that the field equations are satisfied in all reiteration orders up to the $(n-1)$-th. Also, let us assume, as part of the induction hypothesis, that $\underset{(r)}{h}{}^{\mu\nu}{}_{,\nu}$ is a function in which every term contains a $\underset{t}{k}{}^{\cdot\cdot}$, or its derivative with respect to s, multiplied by a function of approximation

(*) See appendix A where $P^{\mu\nu}_{\alpha}$ and $Q^{\mu\nu:\varrho}_{\alpha}$ are defined completely. All that we are interested in here is that $Z^{\mu\nu}$ should be linear in $h^{\mu\nu}{}_{,\nu}$.

order $r - t$. If we call such a function an O_r-function, then we see, from (4.11), that $\underset{(n)}{\Lambda}{}^{\mu\nu}{}_{,\nu}$ is also an O_n-function. From the last two sections, we see that

$$\underset{(n)}{h}{}^{\mu\nu}{}_{,\nu} = +(2\pi)^{-1}\int_V \underset{(n)}{\Lambda}{}^{\mu\nu}{}_{,\nu}(r')\cdot D(r-r')\cdot \mathrm{d}^4 r' + \text{multipoles}\,. \tag{4.12}$$

The first term on the R.H.S. is an O_n-function. In the last section we saw that we could chose the multipoles in such a way that the coefficient of all higher order multipoles were zero and this only involved algebraic processes through (3.25). Also, we can remove the symmetric part of the dipole field from (4.12) as in Sect. 3. The coefficients of the monopoles and the antisymmetric part of the dipole field define the reiteration $\overset{n}{\underset{(n)}{k}}{}^{\cdot\cdot}$, by induction. This shows that the n'th order reiteration field will satisfy the induction hypothesis, *i.e.* it will be an O_n-function, provided that all previous reiteration field do. Consequently, we have proved by induction that the divergence of the reiteration field will be an O_n-function.

It might be thought that our proof depends on the way that we have integrated the field equations, but this is not so. Any solution of the field equations, satisfying the appropriate boundary conditions, outside the surfaces may be obtained from any other by adding a set of multipoles, since the difference satisfies the wave equation. Furthermore, if we use analytic continuation to extend this solution into the interior of the surfaces, we can obtain an arbitrary solution of the field equations.

Theorem 3. Provided that the coordinate conditions are satisfied approximately, in the sense explained above, for all lower order reiteration fields, we may integrate the field equations arbitrarily in the n'th approximation and then satisfy the coordinate conditions to the n'th approximation by the following procedure:

Firstly, we add this $\overset{*}{\underset{n}{h}}{}^{\mu\nu}$ onto the previously calculated reiteration field to give a solution of the n'th order reiteration equations, and then we calculate its divergence, $\overset{*}{\underset{(n)}{h}}{}^{\mu\nu}{}_{,\nu}$. We then use the equations of motion and spin to the $(n-1)$-th approximation to reduce this to a sum of multipoles. The higher order multipoles are eliminated as in (3.25) and the symmetric part of the dipole terms as in (3.20-4). This leaves us with a monopole and the antisymmetric part of the dipole field which define the n'th order reiteration k's

$$\underset{(n)}{h}{}^{\mu\nu}{}_{,\nu} = \sum_p \int_p (\overset{p}{\underset{(n)}{k}}{}^{\mu}\cdot D(r-z) + \overset{p}{\underset{(n)}{k}}{}^{\mu\nu} D(r-z)_{,\nu})\cdot \mathrm{d}s + O_n\text{-function}\,. \tag{4.13}$$

The coordinate conditions will be satisfied to the n'th order, provided that

$$\overset{\mu}{k}_{(n)}, \ \overset{\mu\nu}{k}_{(n)} = 0 \text{ , to the } n\text{'th order.} \tag{4.14}$$

This follows because it implies that $\overset{\mu}{k}_{(r)}$ is zero to the r'th approximation for all $r \leqslant n$.

The interpretation of these equations will depend on the first order solution chosen. Since the source function, $\overset{\mu\nu}{A}_1$, is zero, the field $\overset{\mu\nu}{h}_1$ must satisfy the wave equation. We shall now calculate the most general pole-dipole solution of the wave equation, such that the coordinate conditions are satisfied approximately in the sense above. We shall write the first order solution as

$$\overset{\mu\nu}{h}_1 = \int M^{\mu\nu} \cdot D(r-z) \cdot \mathrm{d}s + \int M^{\mu\nu\varrho} \cdot D(r-z)_{,\varrho} \cdot \mathrm{d}s \, . \tag{4.15}$$

It should be noticed that the dipole and quadrupole moments are related by the equations,

$$\int \eta^{\alpha\beta} \cdot D(r-z)_{,\alpha\beta} \cdot \mathrm{d}s = 0 \, , \tag{4.16}$$

$$\int f(s) \cdot v^{\alpha} \cdot D(r - z_{\alpha\mu}) = \int \dot{f} \cdot D(r-z)_{,\mu} \cdot \mathrm{d}s \, . \tag{4.16'}$$

Equation (4.16)′ follows from (3.16).

We shall look for the most general solution such that

$$\overset{\mu\nu}{h}_1 = \int k^{\mu} \cdot D(r-z) \mathrm{d}s + \int \overset{\mu\nu}{k} \cdot D(r-z)_{,\nu} \cdot \mathrm{d}s \, . \tag{4.17}$$

First of all we observe that we may write $M^{\mu\nu:\varrho}$ as

$$\begin{aligned} M^{\mu\nu:\varrho} &= (M^{\mu\nu:\varrho} - M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho}) + M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho} \, , \\ &= \overset{*}{M}{}^{\mu\nu:\varrho} + M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho} \, . \end{aligned} \tag{4.18}$$

By using (3.16), we may reduce the field corresponding to the second term in (4.18) to a monopole field. Consequently, since the first term satisfies $\overset{*}{M}{}^{\mu\nu:\varrho} \cdot v_{\varrho} = 0$, we may assume that

$$v_{\alpha} M^{\mu\nu:\alpha} = 0 \, , \tag{4.19}$$

without loss of generality.

Also, we have

$$k^{\mu\nu}_{\smile} \cdot v_{\nu} = 0 \,, \tag{4.20}$$

this equation being true for all the approximation and reiteration k's of this section. To see this, let us suppose that we have reduced the divergence of the field tensor in the n'th reiteration to (4.13). We may write the dipole expression as

$$\begin{aligned} k^{\mu\nu}_{\smile} &= (k^{\mu\nu}_{\smile} - k^{\mu\nu}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu}) - (k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu}) + 2k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} \,, \\ &= \overset{*}{k}{}^{\mu\nu}_{\smile} - W^{\mu\nu}_{\smile} + 2k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} \,, \end{aligned} \tag{4.21}$$

where

$$\left\{ \begin{aligned} \overset{*}{k}{}^{\mu\nu}_{\smile} &= k^{\mu\nu}_{\smile} - k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu} \,, \\ W^{\mu\nu}_{\smile} &= k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu} \,. \end{aligned} \right. \tag{4.22}$$

Since $W^{\mu\nu}_{\smile}$ is symmetric in μ and ν, we can remove it from the field by the addition of a monopole field to the field tensor, as in (3.20-4). By the use of (3.16), we can reduce the field corresponding to the third term on the R.H.S. of (4.21) to a monopole field. This leaves us with a similar expression to (4.13) except that $k^{\mu\nu}_{\smile}$ is replaced by $\overset{*}{k}{}^{\mu\nu}_{\smile}$, which satisfies (4.20). This shows that there are only *three independent* differential equations corresponding to the independent components of $k^{\mu\nu}_{\smile}$.

We shall now proced with the calculation of the most general pole-dipole field. From (4.15) we see that

$$\underset{1}{h}{}^{\mu\nu}{}_{,\mu} = \int (M^{\mu\nu}_{\smile} \cdot D(r-z)_{,\nu} + M^{\mu\nu:\varrho}_{\smile} \cdot D(r-z)_{,\varrho\nu}) \cdot \mathrm{d}s \,. \tag{4.23}$$

Equation (4.17) and (4.23) do not imply that $M^{\mu\nu:\varrho}$ is zero, since the multipoles are connected by (4.16) and (4.16)'. Instead, we must have

$$M^{\mu\nu:\varrho}_{\smile} = B^{\mu} \cdot \eta^{\nu\varrho} + C^{\mu\nu} \cdot v^{\varrho} + D^{\mu\varrho} \cdot v^{\nu} + F^{\mu:\nu\varrho}_{\smile} \,, \tag{4.24}$$

where $F^{\mu:\nu\varrho}_{\smile}$ is antisymmetric in ν and ϱ, and the parameters on the right hand side have to be determined. Since the L.H.S. is symmetric in μ and ν, we have

$$F^{\mu:\nu\varrho}_{\smile} - F^{\nu:\mu\varrho}_{\smile} = -B^{\mu} \cdot \eta^{\mu\varrho} + B^{\nu} \cdot \eta^{\mu\varrho} - C^{[\mu\nu]} v^{\varrho} + D^{\nu\varrho} v^{\mu} - D^{\mu\varrho} v^{\nu} \,. \tag{4.25}$$

Therefore, by interchanging the coefficients, we have

$$\begin{aligned} F^{\mu:\nu\varrho}_{\smile} &= \tfrac{1}{2}(F^{\mu:\nu\varrho}_{\smile} - F^{\nu:\mu\varrho}_{\smile}) + \tfrac{1}{2}(F^{\varrho:\nu\mu}_{\smile} - F^{\nu:\varrho\mu}_{\smile}) + (F^{\varrho:\mu\nu}_{\smile} - F^{\mu:\varrho\nu}_{\smile}) = \\ &= B^{\nu}\eta^{\mu\varrho} - B^{\varrho}\eta^{\mu\nu} + C^{[\nu\varrho]}v^{\mu} - C^{[\mu\nu]}v^{\varrho} + C^{[\mu\varrho]}v^{\nu} + D^{(\mu\nu)}v^{\varrho} + D^{[\nu\varrho]}v^{\mu} - D^{(\mu\varrho)}v^{\nu} \,, \end{aligned} \tag{4,26}$$

Consequently, we have that

$$M^{\underset{\sim}{\mu\nu}:\varrho} = (B^{\mu}\eta^{\nu\varrho} - B^{\varrho}\delta^{\mu\nu} + B^{\nu}\eta^{\mu\varrho}) + C^{[\nu\varrho]}v^{\mu} + C^{[\mu\varrho]}v^{\nu} + C^{(\mu\nu)}v^{\varrho} + \\ + D^{[\nu\varrho]}v^{\mu} + D^{[\mu\varrho]}v^{\nu} + D^{(\mu\nu)}v^{\varrho} . \tag{4.27}$$

The first term on the R.H.S. corresponds to the usual allowable coordinate transformations so we shall take $B^{\mu} = 0$. If we substitute

$$E^{\underset{\sim}{\mu\nu}} = C^{[\mu\nu]} + D^{[\mu\nu]}, \qquad G^{\underset{\sim}{\mu\nu}} = C^{(\mu\nu)} + D^{(\mu\nu)}, \tag{4.28}$$

which are obviously antisymmetric and symmetric, respectively, we may write (4,27) as

$$M^{\underset{\sim}{\mu\nu}:\varrho} = v^{\mu}E^{\underset{\sim}{\nu\varrho}} + v^{\nu}E^{\underset{\sim}{\mu\varrho}} + v^{\vartheta}G^{\underset{\sim}{\mu\nu}} . \tag{4.29}$$

If we multiply this equation by v_{ϱ} and use (4.19), we have

$$G^{\underset{\sim}{\mu\nu}} = -v_{\alpha}E^{\underset{\sim}{\mu\alpha}}v^{\nu} - v_{\alpha}E^{\underset{\sim}{\nu\alpha}}v^{\mu} , \tag{4.30}$$

and so we may rewrite the dipole tensor as

$$M^{\underset{\sim}{\mu\nu}:\varrho} = v^{\mu}E^{\underset{\sim}{\nu\varrho}} + v^{\nu}E^{\underset{\sim}{\mu\varrho}} - (v_{\alpha}E^{\underset{\sim}{\mu\alpha}}v^{\nu} + v_{\alpha}E^{\underset{\sim}{\nu\alpha}}v^{\mu})v^{\varrho} , \tag{4.31}$$

$$= 4v^{\mu}S^{\underset{\sim}{\nu\varrho}} + 4v^{\nu}S^{\underset{\sim}{\mu\varrho}} + 4D^{\varrho}v^{\mu}v^{\nu} ,$$

where

$$4S^{\underset{\sim}{\mu\nu}} = E^{\underset{\sim}{\mu\nu}} - E^{\underset{\sim}{\mu\alpha}}v_{\alpha}v^{\nu} + E^{\underset{\sim}{\nu\alpha}}v_{\alpha}v^{\mu} ,$$

$$4D_{\mu} = -2E^{\underset{\sim}{\mu\alpha}}v_{\alpha} .$$

From their definitions, we see that

$$v_{\alpha}S^{\underset{\sim}{\mu\alpha}} = v_{\alpha}D^{\alpha} = 0 . \tag{4.32}$$

If we use (4.31) in (4.15), we have

$$\overset{\mu\nu}{h}_{1\ ,\nu} = \int \left(M^{\underset{\sim}{\varrho\nu}} + 4\frac{\mathrm{d}}{\mathrm{d}s}(D^{\nu}v^{\mu}) + 4\dot{S}^{\underset{\sim}{\mu\nu}} \right) D(r-z)_{,\nu}\,\mathrm{d}s . \tag{4.33}$$

Since this must be equal to the R.H.S. of (4.17),

$$M^{\underset{\sim}{\mu\nu}} + \frac{\mathrm{d}}{\mathrm{d}s}(4D^{\nu}v^{\mu}) + 4\dot{S}^{\underset{\sim}{\mu\nu}} = k^{\underset{\sim}{\mu\nu}} + H^{\mu}v^{\nu}, \tag{4.34}$$

where H^μ is an arbitrary parameter. Taking the antisymmetric part of this equation,

$$2\frac{\mathrm{d}}{\mathrm{d}s}(D^\nu v^\mu - D^\mu v^\nu) + 4\dot{S}^{\mu\nu} + \tfrac{1}{2}(H^\nu v^\mu - H^\mu v^\nu) = k^{\underset{\sim}{\mu\nu}}, \tag{4.35}$$

and multiplying by v_ν,

$$\tfrac{1}{2}\cdot H^\mu - \tfrac{1}{2}\cdot H^\alpha v_\alpha v^\mu = 4S^{\underset{\sim}{\mu\alpha}} v_\alpha - 2\dot{D}^\mu, \tag{4.36}$$

$$k^{\underset{\sim}{\mu\nu}} = 4\dot{S}^{\underset{\sim}{\mu\nu}} - 4(S^{\underset{\sim}{\mu\alpha}} v_\alpha v^\nu - S^{\underset{\sim}{\nu\alpha}} v_\alpha v^\mu) + 2(D^\nu \dot{v}^\mu - D^\mu \dot{v}^\nu).$$

Consequently,

$$H^\mu = mv^\mu + 8\dot{S}^{\underset{\sim}{\mu\alpha}} v_\alpha - 4\dot{D}^\mu, \tag{4.37}$$

and so

$$M^{\underset{\sim}{\mu\nu}} = 4mv^\mu v^\nu + 4(\dot{S} v_\alpha v^\nu + \dot{S}^{\underset{\sim}{\nu\alpha}} v_\alpha v^\mu) - 4(\dot{D}^\mu v^\nu + D^\nu v^\mu) - 2(D^\mu \dot{v}^\nu + D^\nu \dot{v}^\mu), \tag{4.38}$$

where m is an arbitrary parameter: $4m = H^\alpha v_\alpha$.

This shows that we can characterise the most general pole-dipole solution in the first approximation, by a mass parameter, a spin tensor, and the dipole moment of the rest mass, satisfying (4.32). This corresponds to the results of PAPAPETROU for a test body, though his results are slightly different in form because he defined the particle moments differently.

We have for the first order field,

$$\underset{1}{h}^{\mu\nu} = \int (4mv^\mu v^\nu - 4(\dot{D}^\mu v^\nu + \dot{D}^\nu v^\mu) - 2(D^\mu \dot{v} + D^\nu \dot{v}^\mu) - \tag{4.39}$$

$$- 4S^{\mu\alpha}\dot{v}_\alpha v^\nu - 4S^{\nu\alpha}\dot{v}_\alpha v_\pi)\, D(r-z)\,\mathrm{d}s + 4\int (S^{\mu\varrho} v^\nu + S^{\nu\varrho} v^\mu + D^\varrho v^\mu v^\nu) D(r-z)_{,\partial}\,\mathrm{d}s,$$

We have introduced the factor of 4 so that the gravitational constant is unity. The particle tensors must satisfy (4.32).

If we take the divergence of (4.39), we verify that (4.17) holds with the first order reiteration k's as follows:

$$\underset{1}{k}^\mu = 4\frac{\mathrm{d}}{\mathrm{d}s}(mv^\mu) - 4\ddot{D}^\mu - 8\frac{\mathrm{d}}{\mathrm{d}s}(S^{\underset{\sim}{\mu\alpha}}\dot{v}^\alpha), \tag{4.40}$$

$$\underset{1}{k}^{\underset{\sim}{\mu\nu}} = 4\frac{\mathrm{d}}{\mathrm{d}s}(S^{\underset{\sim}{\mu\nu}}) + 4(S^{\underset{\sim}{\mu\varrho}}\dot{v}_\varrho v^\nu - S^{\underset{\sim}{\nu\varrho}}\dot{v}_\varrho v^\mu) + 2(D^\nu \dot{v}^\mu - D^\mu \dot{v}^\nu). \tag{4.41}$$

where it will be observed that $k_1^{\mu\nu}$ satisfies (4.20), and so has only three independent components. In the previous section we were only interested in showing that the field equations and coordinate conditions could be satisfied in each approximation so we did not reduce the $k_n^{[\mu\nu]}$ to functions satisfying (4.20). In this section we are interested in the physical interpretation of the results. The reason that we have been able to satisfy (4.20) is that the multipoles are not independent, but are related by (4.16) and (4.16)'. This leaves only three independent spin *differential* equations.

Let us now suppose that we have found the reiateration k's to the n'th order. When we equate them to zero, and use (4.40) and (4.41),

$$4\frac{\mathrm{d}}{\mathrm{d}s}(mv^\mu) - 4\ddot{D}^\mu - 8\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\alpha}\dot{v}_\alpha) = -\sum_{r=2} k_r, \tag{4.42}$$

$$4\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\nu}) + 4(S^{\mu\alpha}\dot{v}_\alpha v^\nu - S^{\nu\alpha}\dot{v}_\alpha v^\mu) + 2(D^\nu\dot{v}^\mu - D^\mu\dot{v}^\nu) = -\sum_{r=2}^{n} k_r^{\mu\nu}, \tag{4.43}$$

where

$$S^{\mu\nu}v_\nu = D^\mu v_\mu = 0. \tag{4.44}$$

If we multiply (4.42) by v_μ and use the differential of (4.44) and also the equation, $\dot{v}_\alpha v^\alpha = 0$, we have

$$4\dot{m} - 4\ddot{D}^\mu v_\mu + 8v_\mu\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_\alpha) = 4\dot{m} - 4\ddot{D}^\alpha v_\alpha + 8\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_\alpha v_\mu) - 8S^{\mu\alpha}\dot{v}_\mu\dot{v}_\alpha, \tag{4.45}$$

$$= -\sum_{r=2} k_r^\alpha v_\mu,$$

and therefore

$$4\dot{m} = 4\ddot{D}^\alpha v_\alpha - \sum_{r=2}^{n} k^\alpha v_\alpha. \tag{4.45}$$

which is the equation of mass or energy. The remaining three equations of (4.42) give the equations of motion, while (4.43) correspond to the classical equations of angular momentum.

These equations are equivalent to those obtained by PAPAPETROU ([6]) for spinning test particles in general relativity. They are also equivalent to those derived by MATHISSON ([7]), LUBANSKI ([8]), and HÖNL and PAPAPETROU ([9,10]) for

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([5]) A. PAPAPETROU: *Proc. Roy. Soc.*, **209**, 248 (1951).
([6]) M. MATHISSON: *Acta Phys. Polon.*, **6**, 167 (1937).
([7]) H. LUBANSKI: *Acta Phys. Polon.*, **6**, 356 (1937).
([8]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **112**, 512 (1939).
([9]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **114**, 478 (1939).
([10]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **116**, 153 (1940).

spinning test particles in special relativity. What is interesting is that they should arise from the structure of the field equations outside the particles, rather than the precise form of the assumed energy momentum tensor. In a later paper, we shall calculate the equations of motion and spin for a particle in an external field arising from a set of particels.

Also, we shall show that essentially the same equations arise in the quasi-static approximation. The analysis is slightly different in that it is impossible to satisfy the coordinate conditions *exactly* in the n'th approximation unless certain differential equations are satisfied by the particle parameters *to the previous approximations*. As we have seen, the differential equations arising in the n'th covariant approximation involve the particle parameters of the same order. The reason for this is that derivatives with respect to the non-covariant time coordinate in the quasi-static approximation are considered to raise the approximation order of the function differentiated.

It will be noticed that (4.42-3) are inconsistent if we equate the spin tensor to zero. This is also true of the quasi-static approximation. As is to be expected, a dipole particle in an external field will start to spin under the applied couple. It is not possible, in general, to satisfy the coordinate conditions exactly in each approximation by introducting a dipole moment, unless a corresponding spin tensor is also introduced. In the quasi-static approximation there are three extra surface integral conditions to be satisfied as well as those found by EINSTEIN and INFELD. These correspond to the equations of spin.

APPENDIX

Fock has show that, if we introduce the following functions,

$$\pi^{\mu,\alpha\beta} = \frac{1}{2g}\left(\mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}{}_{,\nu} + \mathfrak{g}^{\beta\nu}\mathfrak{g}^{\mu\alpha}{}_{,\nu} - \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\beta\alpha}{}_{,\nu}\right),$$

$$\pi^{\mu\nu}{}_{\lambda} = g_{\mu\alpha}g_{\nu\beta}\pi^{\lambda,\alpha\beta},$$

$$y_{\mu} = (\log(-g)^{\frac{1}{2}})_{,\mu}, \qquad y^{\mu} = g^{\mu\alpha}{}_{,\alpha},$$

$$\Gamma^{\mu} = -(-g)^{-\frac{1}{2}}\mathfrak{g}^{\mu\varrho}{}_{,\varrho},$$

$$\Gamma_{\mu\nu} = \tfrac{1}{2}\left(g_{\mu\alpha}\Gamma^{\alpha}{}_{,\nu} + g_{\nu\alpha}\Gamma^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\Gamma^{\alpha}\right),$$

$$\Gamma^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\Gamma_{\alpha\beta},$$

the Einstein tensor-density may be reduced to the following,

$$\mathfrak{G}^{\mu\nu} = -\tfrac{1}{2}(-g)^{-\frac{1}{2}}\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu}{}_{,\alpha\beta} + \pi^{\mu,\alpha\beta}\pi_{\alpha\beta}{}^{\nu}(-g)^{-\frac{1}{2}} + \tfrac{1}{4}\mathfrak{g}^{\mu\nu}y_{\alpha}y^{\alpha} - \tfrac{1}{2}(-g)^{\frac{1}{2}}y^{\mu}y^{\nu} -$$

$$-\tfrac{1}{4}(-g)^{-\frac{1}{2}}\cdot\mathfrak{g}^{\mu\nu}\pi_{\alpha\beta}{}^{\varrho}\cdot\mathfrak{g}^{\alpha\beta}{}_{,\varrho} - (-g)^{\frac{1}{2}}\Gamma^{\mu\nu} - \tfrac{1}{2}(-g)^{\frac{1}{2}}(\Gamma^{\mu}y^{\nu} + \Gamma^{\nu}y^{\mu}) + \tfrac{1}{2}\mathfrak{g}^{\mu\nu}(g^{\alpha\beta}\Gamma_{\alpha\beta} + \Gamma^{\alpha}y_{\alpha}).$$

For our purposes, it is sufficient that it should be possible to write this as

$$2\mathfrak{G}^{\mu\nu} = -h^{\mu\nu} + 2\Lambda^{\mu\nu} + Z^{\mu\nu},$$

where

$$Z^{\mu\nu} = h^{\alpha\beta}{}_{,\beta}P^{\mu\nu}_{\mu} + h^{\alpha\beta}{}_{,\beta\varrho}Q^{\mu\nu:\varrho}_{\alpha}.$$

$Z^{\mu\nu}$ consists of the last three terms in the expression of $\mathfrak{G}^{\mu\nu}$. This is so because the last three terms are all linear in Γ^{α}, or its derivative.

RIASSUNTO (*)

Si è analizzato il metodo d'approssimazione covariante di Lorentz per il campo esterno a un sistema di particelle localizzate. Si è trovato che oltre le solite equazioni del moto e dell'energia ottenute da Einstein, Infeld e Hoffman per l'approssimazione quasi statica ne esistono altre tre, le equazioni dello spin, che debbono essere soddisfatte dalle costanti strutturali di ogni particella. Tali equazioni appaiono anche nella approssimazione quasi statica come condizioni degli integrali di superficie. Si dimostra inoltre che non è necessario sviluppare in serie le costanti di massa, dipolo o spin, essendo le costanti introdotte nella prima approssimazione quelle delle particelle fisiche. Solo le equazioni differenziali soddisfatte da queste costanti cambiano nelle approssimazioni degli ordini superiori.

(*) *Traduzione a cura della Redazione.*

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On the Lorentz-Covariant Approximation Method in General Relativity.

II. - Second Approximation.

R. P. Kerr

Trinity College - Cambridge

(ricevuto il 4 Settembre 1958)

Summary. — In this paper we have calculated the equations of motion and of spin to the second approximation by the method derived by the author in the first paper of this series. The results are completely analogous to those derived by Papapetrou [1] for a pole-dipole particle. However, following the spirit of Einstein, Infeld and Hoffman, we have derived them from the field outside the particle whereas Papapetrou derived them from an energy momentum tensor.

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1. – Derivation of equations.

In the first paper of this series (*) we considered the fast particle approximation method in general relativity and showed how the equations of motion and of spin could be derived by successive approximation. In this paper we shall calculate these particle equations to the second order for a particle with non-zero mass and spin tensor, but whose higher order moments are all zero.

Since we have considered the theory more generally in (I) we shall only give a very brief outline here. As usual, we expand the fundamental contravariant tensor density as a series,

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + 4U^{\mu\nu} + 4V^{\mu\nu} + \dots , \tag{1.1}$$

(*) R. P. Kerr: *Nuovo Cimento*, **13**, 469 (1959) hereafter referred to as I.

where $\eta^{\mu\nu}$ is the Galilean metric tensor with signature $(+,-,-,-)$, and $U^{\mu\nu}$ and $V^{\mu\nu}$ are of the first and second order, respectively. The reduced field equations may be written as

$$\Box U^{\mu\nu} = 0\,, \tag{1.2}$$

$$\Box V^{\mu\nu} = \tfrac{1}{2}\overset{2}{\Delta}{}^{\mu\nu}\,, \tag{1.3}$$

where $\overset{2}{\Delta}{}^{\mu\nu}$ is given by

$$\begin{aligned}\tfrac{1}{2}\overset{2}{\Delta}{}^{\mu\nu} = &-4U^{\alpha\beta}U^{\mu\nu}{}_{,\alpha\beta} + 4U^{\mu\alpha}{}_{,\beta}U^{\nu}{}_{\alpha}{}^{,\beta} + 4U^{\mu\alpha}{}_{,\beta}U^{\nu\beta}{}_{,\alpha} - 4U^{\mu\alpha,\beta}U_{\alpha\beta}{}^{,\nu} - \\ &-4U^{\nu\alpha,\beta}U_{\alpha\beta}{}^{,\mu} + 2U^{\alpha\beta,\mu}U_{\alpha\beta}{}^{,\nu} - U_{\alpha}{}^{\alpha,\mu}U_{\beta}{}^{\beta,\nu} + \eta^{\mu\nu}(2U^{\alpha\beta}{}_{,\varrho}U^{\varrho}{}_{\alpha,\beta} - U^{\alpha\beta\varrho}U_{\alpha\beta;\varrho} + \tfrac{1}{2}U_{\alpha}{}^{\alpha,\varrho}U_{\beta}{}^{\beta}{}_{,\varrho})\,.\end{aligned} \tag{1.4}$$

(*)

This may be proved by expanding out the reduced field equations of Appendix A in (I).

From equation (4.39) of (I), we see that the solution of (1.2) for a simple spinning particle is

$$U^{\mu\nu} = \overset{s}{U}{}^{\mu\nu} + \overset{*}{U}{}^{\mu\nu}\,, \tag{1.5}$$

where

$$\begin{aligned}\overset{s}{U}{}^{\mu\nu} = &\int (mv^{\mu}v^{\nu} + \dot{S}^{\mu\alpha}v_{\alpha}v^{\nu} + \dot{S}^{\nu\alpha}v_{\alpha}v^{\alpha})\,D(r-z)\,\mathrm{d}s + \\ &+ \int (S^{\mu\varrho}v^{\nu} + S^{\nu\varrho}v^{\mu})\,D(r-z)_{,\varrho}\,\mathrm{d}s\,,\end{aligned} \tag{1.6}$$

and

$$v_{\alpha}S^{\mu\alpha} = 0\,. \tag{1.7}$$

$S^{\mu\nu}$ is the antisymmetric spin tensor. $\overset{*}{U}{}^{\mu\nu}$ is the external potential, which is non-singular at the particle considered, and satisfies

$$\eta^{\alpha\beta}\overset{*}{U}{}^{\mu\nu}{}_{,\alpha\beta} = \overset{*}{U}{}^{\mu\nu}{}_{,\nu} = 0\,, \tag{1.8}$$

to the second approximation. As was shown in (I), the equations of motion and spin are, to the first approximation,

$$\overset{1}{k}{}^{\mu} = 4\frac{\mathrm{d}}{\mathrm{d}s}(mv^{\mu}) + 8\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_{\alpha}) = 0\,, \tag{1.9}$$

$$\overset{1}{k}{}^{\mu\nu} = 4\dot{S}^{\mu\nu} - 4\dot{S}^{\mu\alpha}v_{\alpha}v^{\nu} + 4\dot{S}^{\nu\alpha}v_{\alpha}v^{\mu} = 0\,. \tag{1.10}$$

(*) In this paper, we shall raise and lower the indices of $U^{\mu\nu}$, v^{μ}, etc., with the Galilean tensor, $\eta^{\alpha\beta}$.

This means that the mass, velocity, and spin tensor are all constants to the first approximation.

Following (I), we take as a particular solution of (1.3)

$$\overset{*}{V}{}^{\mu\nu} = (8\pi)^{-1}\int \underset{2}{\Lambda}{}^{\mu\nu}(r')\cdot D(r-r')\cdot \mathrm{d}^4 r' \,, \tag{1.11}$$

where V denotes the 4-volume outside the surface, S, surrounding the particle. From the analysis of (I), we know that we can add harmonic functions to this particular solution, so that

$$4(U^{\mu\nu}+V^{\mu\nu})_{,\nu} = \int (\underset{1}{k}{}^{\mu} + \overset{*}{\underset{2}{k}}{}^{\mu})\cdot D(r-z)\,\mathrm{d}s + \int (\underset{1}{k}{}^{\mu\nu} + \overset{*}{\underset{2}{k}}{}^{\mu:\nu})\cdot D(r-z)_{,\varrho}\,\mathrm{d}s \,, \tag{1.12}$$

where

$$\overset{*}{\underset{2}{k}}{}^{\mu} = (2\pi)^{-1}\int \underset{2}{\Lambda}((1-(\dot{v},\gamma))\cdot \mathrm{d}^2 S_\nu \,, \tag{1.13}$$

$$\overset{*}{\underset{2}{k}}{}^{\mu:\nu} = -(2\pi)^{-1}\int \underset{2}{\Lambda}{}^{\mu\varrho}\cdot\gamma^\nu(1-(\dot{v},\gamma))\cdot \mathrm{d}^2 S_\varrho \,. \tag{1.14}$$

If we express

$$\overset{*}{\underset{2}{k}}{}^{\mu:\nu} = (\overset{*}{\underset{2}{k}}{}^{[\mu:\nu]} + \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu - \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\nu}v_\alpha v^\mu) + (\overset{*}{\underset{2}{k}}{}^{(\mu:\nu)} + \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu + \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\nu}v_\alpha v^\mu) - \overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu \,, \tag{1.15}$$

then we can remove the second term in (1.15) from (1.12) by adding a symmetric monopole solution,

$$-\frac{1}{4}\int\left(\overset{*}{\underset{2}{k}}{}^{(\mu:\nu)} + \frac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu + \frac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\nu}v_\alpha v^\mu\right) D(r-z)\cdot \mathrm{d}s \,,$$

to $V^{\mu\nu}$. This leaves us with

$$4(U^{\mu\nu}+V^{\mu\nu})_{,\nu} = \int (\underset{1}{k}{}^{\mu} + \overset{*}{\underset{2}{k}}{}^{\mu})D(r-z)\,\mathrm{d}s + \int (\underset{1}{k}{}^{\mu\nu} + \overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu)D_{,\nu}\,\mathrm{d}s = \tag{1.16}$$

$$= \int (\underset{1}{k}{}^{\mu} + \underset{2}{k}{}^{\mu})D(r-z)\,\mathrm{d}s + \int (\underset{1}{k}{}^{\mu\varrho} + \underset{2}{k}{}^{\mu\varrho})D(r-z)_{,\varrho}\,\mathrm{d}s \,,$$

where

$$\underset{(2)}{k}{}^{\mu} = \underset{1}{k}{}^{\mu} + \underset{2}{k}{}^{\mu} = \underset{1}{k}{}^{\mu} + \overset{*}{\underset{2}{k}}{}^{\mu} - \frac{\mathrm{d}}{\mathrm{d}s}(\overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha), \tag{1.17}$$

$$\underset{(2)}{k}{}^{\mu\nu} = \underset{1}{k}{}^{\mu\nu} + \underset{2}{k}{}^{\mu\nu} = \underset{1}{k}{}^{\mu\nu} + \overset{*}{\underset{2}{k}}{}^{[\mu:\nu]} + \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\mu}v_\alpha v^\nu - \tfrac{1}{2}\overset{*}{\underset{2}{k}}{}^{\alpha:\nu}v_\alpha v^\mu \,. \tag{1.18}$$

When we equate these to zero, we obtain the equations of motion and spin to the second approximation. From (1.18), we verify that

$$v_\alpha k_2^{\mu\alpha} = 0 \,. \tag{1.19}$$

The surface integrals in (1.13-4) are taken over the two dimensional surfaces defined by the conditions,

$$s(r) = \text{const} \,, \qquad \eta_{\alpha\beta}\gamma^\alpha\gamma^\beta = -a^2 \,, \tag{1.20}$$

where a is the radius of the surface, S. As explained in (I), $\gamma^\lambda(r)$ and $s(r)$ are defined by the equations,

$$\gamma^\alpha(r) = x^\alpha - z^\alpha(s(r)) \,, \tag{1.21}$$

$$\gamma^\alpha(r) v_\alpha(s(r)) = 0 \,. \tag{1.22}$$

2. – The calculation of the surface integrals.

We shall now calculate the surface integrals, (1.13-4). Since we are only calculating the equations of motion and spin to the second approximation, we may use the equations of motion and spin to the first order to simplify these integrals. We shall discuss this more fully in the section on the equations of motion. From (1.9-10), we see that the surface integrals are to be calculated for m, v^μ, and $S^{\mu\nu}$ all constant. For such constant motion, we have

$$\int D(r-z)\,\mathrm{d}s = \gamma^{-1} \,, \tag{2.1}$$

where

$$\eta_{\alpha\beta}\gamma^\alpha\gamma^\beta = -\gamma^2 \,. \tag{2.2}$$

This is the usual expression for the potential of a non-accelerated source and may be verified easily in the rest system of the particle.

From (1.6),

$$\overset{*}{U}{}^{\mu\nu} = mv^\mu v^\nu \gamma^{-1} + (v^\mu S^{\nu\varrho} + v^\nu S^{\mu\varrho})(\gamma^{-1}) \,. \tag{2.3}$$

If we differentiate (1.21-2) with respect to x^β, we have

$$\gamma^\alpha{}_{,\beta} = \delta^\alpha_\beta - v^\alpha \frac{\partial s}{\partial x^\beta} \,,$$

$$\left(\delta^\alpha_\beta - v^\alpha \frac{\partial s}{\partial x^\beta}\right) v_\alpha = 0 \,,$$

and so

$$\gamma^{\alpha}_{\ ,\beta} = \delta^{\alpha}_{\beta} - v^{\alpha}v_{\beta}\,. \tag{2.4}$$

Consequently, we have

$$(\gamma^{-1})_{,\varrho} = \frac{\gamma_{\alpha}\gamma^{\alpha}_{\ ,\varrho}}{\gamma^3} = \frac{\gamma_{\varrho}}{\gamma^3}\,, \tag{2.5}$$

and so

$$\overset{s}{U}{}^{\mu\nu} = mv^{\mu}v^{\nu}\gamma^{-1} + (v^{\mu}S^{\nu\varrho} + v^{\nu}S^{\mu\varrho})\frac{\gamma_{\beta}}{\gamma^3}\,. \tag{2.6}$$

We shall need two further expressions,

$$\overset{s}{U}{}^{\mu\nu}_{\ ,\alpha\beta} = mv^{\mu}v^{\nu}\frac{\gamma_{\alpha}}{\gamma^3} + (v^{\mu}S^{\nu\varrho} + v^{\nu}S^{\mu\varrho})\left(\frac{3\gamma_{\varrho}\gamma_{\alpha}}{\gamma^5} + \frac{\eta_{\varrho\alpha}}{\gamma^3}\right), \tag{2.7}$$

$$\begin{aligned}\overset{s}{U}{}^{\mu\nu}_{\ ,\alpha\beta} = {} & mv^{\mu}v^{\nu}\left(3\frac{\gamma_{\alpha}\gamma_{\beta}}{\gamma^5} + \frac{\eta_{\alpha\beta} - v_{\alpha}v_{\beta}}{\gamma^3}\right) + \\ & + (v^{\mu}S^{\nu\varrho} + v^{\mu}S^{\nu\varrho})\left(15\cdot\frac{\gamma_{\varrho}\gamma_{\alpha}\gamma_{\beta}}{\gamma^7} + 3\frac{\gamma_{\alpha}\eta_{\beta\varrho} + \gamma_{\beta}\eta_{\alpha\varrho} + \gamma_{\varrho}\eta_{\alpha\beta} - \gamma_{\varrho}v_{\alpha}v_{\beta}}{\gamma^5}\right)\end{aligned} \tag{2.8}$$

where we have used (1.7) to simplify the results.

We shall also need the following surface integrals,

$$(4\pi)^{-1}\int \mathrm{d}^2S_{\nu} = 0\,, \tag{2.9}$$

$$(4\pi)^{-1}\int \gamma_{\mu}\cdot\mathrm{d}^2S_{\nu} = -\frac{a^3}{3}(\eta_{\mu\nu} - v_{\mu}v_{\nu})\,, \tag{2.10}$$

$$(4\pi)^{-1}\int \gamma_{\lambda}\gamma_{\mu}\cdot\mathrm{d}^2S_{\nu} = 0\,, \tag{2.11}$$

$$\begin{aligned}(4\pi)^{-1}\int \gamma_{\varrho}\gamma_{\lambda}\gamma_{\mu}\cdot\mathrm{d}^2S_{\nu} = {} & \frac{a^5}{15}(\eta_{\varrho\lambda}\eta_{\mu\nu} + \eta_{\varrho\mu}\eta_{\lambda\nu} + \eta_{\varrho\nu}\eta_{\lambda\mu} - \eta_{\varrho\lambda}v_{\mu}v_{\nu} - \eta_{\varrho\mu}v_{\lambda}v_{\nu} - \\ & - \eta_{\varrho\nu}v_{\lambda}v_{\mu} - \eta_{\lambda\mu}v_{\nu}v_{\varrho} - \eta_{\lambda\nu}v_{\mu}v_{\varrho} - \eta_{\mu\nu}v_{\varrho}v_{\lambda} + 3v_{\mu}v_{\lambda}v_{\varrho}v_{\nu})\,.\end{aligned} \tag{2.12}$$

Since both sides of these equations are tensors, they only need to be verified in the rest system of the particle. The verification is trivial because they are then equivalent to the surface integrals of Einstein, Infeld and Hoffman.

We shall now calculate the self terms in $\overset{s}{k}{}^{\mu}_{2}$. The contribution from the

first term in (1.4) is

$$(2.13)\qquad -16mv^{\alpha}v^{\beta}(v^{\mu}S^{\nu\varrho}+v^{\nu}S^{\mu\varrho})(4\pi)^{-1}\int\gamma^{-1}(\gamma^{-1})_{,\varrho\alpha\beta}\mathrm{d}^2S_{\nu}-$$
$$-16mv^{\mu}v^{\nu}(v^{\alpha}S^{\beta\varrho}+v^{\beta}S^{\alpha\varrho})(4\pi)^{-1}\int(\gamma^{-1})_{,\varrho}(\gamma^{-1})_{,\alpha\beta}\mathrm{d}^2S_{\nu}=0\,.$$

The first term is zero because

$$(2.14)\qquad v^{\alpha}F(\gamma^{\mu})_{,\alpha}=0\,,$$

for any function of γ^{μ}, while the second term is zero because

$$(2.15)\qquad v^{\nu}\mathrm{d}^2S_{\nu}=0\,.$$

The terms containing the product of two masses or two spin constants are zero because the corresponding surface integrals contain an even number of γ^{α} (see (2.9-11)). Similarly, if we calculate the rest of the self terms in $\overset{*}{k}{}^{\mu}_{2}$, we find that they are all zero. On the other hand, there are finite contributions from the self terms in $\overset{*}{k}{}^{\mu\nu}_{2}$. However, they are symmetric in μ and ν, and so they do not contribute to the equations of motion or spin.

We shall now calculate the terms from the external field, $\overset{*}{U}{}^{\mu\nu}$. Since this is regular near the world line, we may expand it about $z^{\mu}(s)$,

$$(2.16)\qquad \overset{*}{U}{}^{\mu\nu}(\gamma,s)=\overline{U}^{\mu\nu}+\overline{U}^{\mu\nu}{}_{,\alpha}\gamma^{\alpha}+\tfrac{1}{2}\overline{U}^{\mu\nu}{}_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}+\dots\,,$$

where

$$\overline{U}^{\mu\nu}{}_{,\alpha\dots}=\overline{U}^{\mu\nu}{}_{,\alpha\dots}(z^{\lambda}(s))\,.$$

We shall neglect all terms in the equations of motion which are proportional to a^2, a^4, etc., which corresponds to taking the limit as the radius of the surface, S, tends to zero. We shall discuss this more fully in Sect. 3.

The contribution to $\overset{*}{k}{}^{\mu}_{2}$ from the first term in (1.4) is

$$-16mv^{\mu}v^{\nu}(4\pi)^{-1}\int(\gamma^{-1})_{,\alpha\beta}(\overline{U}^{\alpha\beta}+\overline{U}^{\alpha\beta}{}_{,\lambda}\gamma^{\lambda}+\dots)\mathrm{d}^2S_{\nu}-$$
$$-16(v^{\alpha}S^{\beta\varrho}+v^{\beta}S^{\alpha\varrho})(4\pi)^{-1}\int(\gamma^{-1})_{,\varrho}(\overline{U}^{\mu\nu}{}_{,\alpha\beta}+\dots)\mathrm{d}^2S_{\nu}-$$
$$-16(v^{\mu}S^{\nu\varrho}+v^{\nu}S^{\mu\varrho})(4\pi)^{-1}\int(\gamma^{-1})_{,\varrho\alpha\beta}(\overline{U}^{\alpha\beta}+\overline{U}^{\alpha\beta}{}_{,\lambda}\cdot\gamma^{\lambda}+\frac{1}{2}\overline{U}^{\alpha\beta}{}_{,\lambda\delta}\gamma^{\lambda}\gamma^{\delta}+\dots)\,\mathrm{d}^2S_{\nu}\,.$$

The first term is zero because $v^{\nu}\mathrm{d}^2S_{\nu}=0$. The second term gives

$$(2.17)\qquad -\frac{32}{3}S^{\alpha\beta}\overline{U}^{\mu}{}_{\alpha,\beta 0}\,,$$

where we use the notation,

$$\overline{U}^{\alpha}{}_{0,\beta\ldots} = v^{\varrho}\overline{U}^{\alpha}{}_{\varrho,\beta\ldots}, \qquad \overline{U}^{\alpha\beta}{}_{,0\ldots} = v^{\varrho}\overline{U}^{\alpha\beta}{}_{,\varrho\ldots}, \quad \text{etc.} \tag{2.18}$$

If we use (2.8) and (2.15) and the fact that $S^{\nu\varrho}$ is antisymmetric, we may reduce the last term to

$$-16v^{\mu}S^{\nu\varrho}\overline{U}^{\alpha\beta}(4\pi)^{-1}\int(6\gamma_{\alpha}\eta_{\beta\varrho})\gamma^{-5}\mathrm{d}^2S_{\nu} - \tag{2.19}$$

$$-8v^{\mu}S^{\nu\varrho}\overline{U}^{\alpha\beta}{}_{,\lambda\delta}(4\pi)^{-1}\int(6\gamma_{\alpha}\eta_{\beta\varrho})\,\frac{\gamma^{\lambda}\gamma^{\delta}}{\gamma^{5}}\,\mathrm{d}^2S_{\nu} = -\frac{32}{5}\cdot v^{\mu}S^{\alpha\beta}\cdot\overline{U}_{0\alpha,\beta 0},$$

where we have used (1.7), (1.8), (21.2) and the antisymmetry of $S^{\nu\varrho}$.

(2.17) and (2.19) give the contributions to $\overset{*}{k}{}^{\mu}_{2}$ from the first term in (1.4). We shall not calculate the contributions from the other terms here as it would take too long. Furthermore, it will be seen that these calculations are purely algebraic, and so we will only give the final results,

$$\frac{1}{4}\overset{*}{k}{}^{\mu}_{2} = +\frac{4}{3}mv^{\mu}(\overline{U}_{00,0} - \overline{U}^{\alpha}{}_{\alpha,0}) + 4m\overline{U}^{\mu}{}_{0,0} - 2m\overline{U}_{00}{}^{,\mu} + m\overline{U}_{\alpha}{}^{\alpha,\mu} - \tag{2.20}$$

$$-\frac{8}{5}S^{\mu\alpha}\overline{U}_{0\alpha,00} + \frac{28}{15}S^{\mu\alpha}\overline{U}_{00,\alpha 0} + \frac{4}{5}S^{\mu\alpha}\overline{U}^{\varrho}{}_{\varrho,0\alpha} -$$

$$-\frac{8}{3}v^{\mu}S^{\alpha\beta}\overline{U}_{0\alpha,\beta 0} - 4S^{\alpha\beta}\overline{U}^{\mu}{}_{\alpha,\beta 0} + 4S^{\mu\alpha}\overline{U}_{0\alpha,\beta}{}^{,\mu},$$

$$\frac{1}{4}\cdot\overset{*}{k}{}^{\mu;\nu}_{2} = \frac{8}{3}(v^{\mu}v^{\nu} - \eta^{\mu\nu})S^{\alpha\beta}\overline{U}_{0\alpha,\beta} - \frac{4}{5}v^{\mu}S^{\nu\alpha}\overline{U}_{00,\alpha} - \frac{8}{3}v^{\nu}S^{\mu\alpha}\overline{U}_{00,\alpha} - \tag{2.21}$$

$$-\frac{8}{5}(v^{\mu}S^{\nu\alpha} - v^{\nu}S^{\mu\alpha})\overline{U}_{0\alpha,0} + \frac{4}{5}v^{\mu}S^{\nu\alpha}\overline{U}^{\varrho}{}_{\varrho,\alpha} - \frac{8}{3}S^{\mu\alpha}\overline{U}_{0\alpha}{}^{,\nu} + \frac{4}{3}S^{\nu\alpha}\overline{U}_{0\alpha}{}^{,\mu} +$$

$$+\frac{16}{15}S^{\mu\alpha}\overline{U}^{\nu}{}_{\alpha,0} - \frac{4}{3}S^{\nu\alpha}\overline{U}^{\mu}{}_{\alpha,0} + \frac{8}{3}S^{\mu\alpha}\overline{U}^{\nu}{}_{0,\alpha} - \frac{4}{3}S^{\nu\alpha}U^{\mu}{}_{0,\alpha} + \frac{4}{5}S^{\mu\nu}(\overline{U}_{00,0} - \overline{U}^{\varrho}{}_{\varrho,0}).$$

3. – Equations of motion.

From (2.21), we see that

$$\frac{1}{4}\cdot\overset{*}{k}{}^{\varrho;\mu}_{2}v_{\varrho} = S^{\mu\alpha}\left(\frac{4}{5}\cdot\overline{U}^{\varrho}{}_{\varrho,\alpha} - \frac{8}{5}\cdot\overline{U}_{0\alpha,0} - \frac{32}{15}\cdot\overline{U}_{00,\alpha}\right). \tag{3.1}$$

Since $\dot{S}^{\mu\nu}$ is zero in this approximation, and since

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\bar{U}^{\varrho}{}_{\varrho,\alpha} = v^{\beta}\bar{U}^{\varrho}{}_{\varrho,\alpha\beta} = \bar{U}^{\varrho}{}_{\varrho,\alpha 0}\,, \quad \text{etc.}\,, \tag{3.2}$$

we have

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\,(\overset{*}{k}{}^{\varrho:\mu}_{2}v_{\varrho}) = S^{\mu\alpha}\left(\frac{4}{5}\,\bar{U}^{\varrho}{}_{\varrho,\alpha 0} - \frac{8}{5}\,\bar{U}_{0\alpha,00} - \frac{32}{15}\,\bar{U}_{00,\alpha 0}\right). \tag{3.3}$$

From (2.20) and (3.1), and from the definition of k_{2}^{μ}, *i.e.* (1.17),

$$\begin{aligned}\frac{1}{4}\,k^{\mu}_{2} = \frac{4}{3}\,mv^{\mu}(\bar{U}_{00,0} - \bar{U}^{\alpha}{}_{\alpha,0}) - \frac{8}{3}\,v^{\mu}S^{\alpha\beta}\bar{U}_{0\alpha,\beta 0} + m(4\bar{U}^{\mu}{}_{0,0} - 2\bar{U}_{00}{}^{,\mu} + \bar{U}_{\alpha}{}^{\alpha,\mu}) + \\ + 4S^{\mu\alpha}\bar{U}_{00,\alpha 0} + 4S^{\alpha\beta}(\bar{U}_{0\alpha,\beta}{}^{,\mu} - \bar{U}^{\mu}{}_{\alpha,\beta 0})\,.\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}\frac{1}{4}\,k^{\mu\nu}_{2} = -\frac{4}{5}\,(v^{\mu}S^{\nu\alpha}\bar{U}_{0\alpha,0} - v^{\nu}S^{\mu\alpha}\bar{U}_{0\alpha,0} + S^{\mu\alpha}\bar{U}^{\nu}{}_{\alpha,0} - S^{\nu\alpha}\bar{U}^{\mu}{}_{\alpha,0}) + \\ + 2(v^{\mu}S^{\nu\alpha} - v^{\nu}S^{\mu\alpha})\,\bar{U}_{00,\alpha} + 2S^{\mu\alpha}(\bar{U}^{\nu}{}_{0,\alpha} + \bar{U}^{\nu}{}_{\alpha,0} - \bar{U}_{0\alpha}{}^{,\nu}) - \\ - 2S^{\nu\alpha}(\bar{U}^{\mu}{}_{0,\alpha} + \bar{U}^{\mu}{}_{\alpha,0} - \bar{U}_{0\alpha}{}^{,\mu})\,.\end{aligned} \tag{3.5}$$

The first term in (3.5) may be removed by changing our solution of the field equations. We add on a field like (1.6) with spin constant given by,

$$\overset{*}{S}{}^{\mu\nu}_{2} = -\frac{4}{5}\,(v^{\mu}S^{\nu\alpha}\bar{U}_{0\alpha} - v^{\nu}S^{\mu\alpha}\bar{U}_{0\alpha} + S^{\mu\alpha}\bar{U}^{\nu}{}_{\alpha} - S^{\nu\alpha}\bar{U}^{\mu}{}_{\alpha})\,. \tag{3.6}$$

It will be seen that this is a valid spin tensor since it satisfies (1.7). From (3.2), we see that $\overset{\cdot}{S}{}^{\mu\alpha}_{2}v_{\alpha} = 0$, and for (1.9,10) that this does not change the equations of motion but cancels the first term in (3.5).
Finally, we obtain the equations of spin,

$$\begin{aligned}\dot{S}^{\mu\nu} - \dot{S}^{\mu\alpha}v_{\alpha}v^{\nu} + \dot{S}^{\nu\alpha}v_{\alpha}v^{\mu} = 2S^{\mu\alpha}(v^{\nu}\bar{U}_{00,\alpha} + \bar{U}_{0\alpha}{}^{,\nu} - \bar{U}^{\nu}{}_{0,\alpha} - \bar{U}^{\nu}{}_{\alpha,0}) - \\ - 2S^{\nu\alpha}(v^{\mu}\bar{U}_{00,\alpha} + \bar{U}_{0\alpha}{}^{,\mu} - \bar{U}^{\mu}{}_{0,\alpha} - \bar{U}^{\mu}{}_{\alpha,0})\,.\end{aligned} \tag{3.7}$$

It will be seen that there are only three independent equations since $\overset{\mu\nu}{(2)}k_{\nu} = 0$.

For the particular solution that we have derived, the equations of motion are, to the second order,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\,(mv^{\mu}) = -2\,\frac{\mathrm{d}}{\mathrm{d}s}\,(\dot{S}^{\mu\alpha}v_{\alpha}) + \frac{4}{3}\,mv^{\mu}(\bar{U}^{\alpha}{}_{\alpha,0} - \bar{U}_{00,0}) + \frac{8}{3}\,v^{\mu}S^{\alpha\beta}\bar{U}_{0\alpha,\beta 0} + \\ + m(2\bar{U}_{00}{}^{,\mu} - \bar{U}_{\alpha}{}^{\alpha,\mu} - 4\bar{U}^{\mu}{}_{0,0}) - 4S^{\mu\alpha}\bar{U}_{00,\alpha 0} + 4S^{\alpha\beta}(\bar{U}^{\mu}{}_{\alpha,\beta 0} - \bar{U}_{0\alpha,\beta}{}^{,\mu})\,.\end{aligned} \tag{3.8}$$

If we multiply this equation by v_μ, we obtain the equation of mass,

$$\dot{m} = -2(S^{\dot{\mu}\alpha}\dot{v}_\alpha + \ddot{S}^{\mu\alpha}v_\alpha)\cdot v_\mu + \frac{4}{3}m(\overline{U}^{\alpha}{}_{\alpha,0} - \overline{U}_{00,0})\cdot + \tag{3.9}$$

$$+ \frac{8}{3}S^{\alpha\beta}\overline{U}_{0\alpha,\beta 0} + m(2\overline{U}_{00,0} + \overline{U}_{\alpha}{}^{\alpha}{}_{,0})\,.$$

If we differentiate (1.7), we have

$$S^{\mu\alpha}\dot{v}_\mu = -\dot{S}^{\mu\alpha}v_\mu\,, \tag{3.10}$$

From this and the antisymmetry of the spin tensor, we see that the first term in (3.9) is zero. To the second approximation, we have

$$\dot{m} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{3}m\overline{U}^{\alpha}{}_{\alpha} - \frac{10}{3}m\overline{U}_{00} + \frac{8}{3}S^{\alpha\beta}\overline{U}_{0\alpha,\beta}\right). \tag{3.11}$$

Since this equation is integrable ,we shall redefine the mass to the second approximation by adding to the field, $V^{\mu\nu}$, the function

$$-\int\left(\frac{1}{3}m\overline{U}^{\alpha}{}_{\alpha} - \frac{10}{3}m\overline{U}_{00} + \frac{8}{3}S^{\alpha\beta}\overline{U}_{0\alpha,\beta}\right)v^\mu v^\nu\cdot D(r-z)\cdot \mathrm{d}s\,.$$

This gives the equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}s}(mv^\mu) = -2\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_\alpha) + m(2\overline{U}_{00}{}^{,\mu} - \overline{U}_{\alpha}{}^{\alpha,\mu} - 4\overline{U}^{\mu}{}_{0,0}) + \tag{3.12}$$

$$+ mv^\mu(2\overline{U}_{00,0} + \overline{U}^{\alpha}{}_{\alpha,0}) + 4S^{\alpha\beta}(\overline{U}^{\mu}{}_{\alpha,\beta 0} - \overline{U}_{0\alpha,\beta}{}^{,\mu}) - 4S^{\mu\alpha}\overline{U}^{00,\alpha 0}\,.$$

We have, of course, redefined the mass in such a way that

$$\frac{\mathrm{d}}{\mathrm{d}s}(m) = 0\,, \tag{3.13}$$

to the second approximation.

For the case where the spin tensor is zero, equation (3.12) is the usual geodesic equation for the motion of a test particle in a gravitational field, $\overset{*}{U}{}^{(\nu)}$, to the second approximation. For the higher approximations, of course, the equivalent statement whould have very little meaning, except for a « test particle », since any proof whould depend on the way that the gravitational field was split into an external and a self field. For a spinning particle, the

equations of motion and spin, (3.7) and (3.12), are equivalent to those derived by PAPATREOU from a finite source tensor [1].

In the derivation of these equations, we have neglected all terms which contain the derivatives of any of the particle constants. This eliminates a the radiation forces, since, if they exist, they should be proportional to the square of the acceleration. The reason for neglecting these terms is that they are zero to the second approximation, from the equations of motion and of spin *to the first approximation*. For instance, let us suppose that the equations of mass and motion are, to the second approximation,

$$\dot{m} = b(\dot{v}, \dot{v}) ,$$

$$4m\dot{v}^{\mu} = - \overset{1}{k}{}^{\mu} .$$

It is quite possible that, when we calculate the third order terms in these equations, we should get

$$\dot{m} = b(\dot{v}, \dot{v}) + \tfrac{1}{4} c \dot{v}_{\alpha} \cdot \overset{2}{k}{}^{\alpha} .$$

If so, then the last term is $-c(\dot{v}, \dot{v})$ to the third approximation. There are two possibilities, if this is so; either $(b–c)$ is zero and so the mass is a constant to the third approximation, or else the coefficient of $(\dot{v}, \dot{v})$ is different from b. In both cases we see that the coefficient of the radiation term is not defined by the coefficient in the second approximation. Just because it is non-zero in the second approximation, it does not follow that there is radiation of gravitational mass. Actually, it is necessary to go to the *fourth* approximation to determine whether $b(\dot{v}, \dot{v})$ does represent radiation. The reason for this is that there might be a further term in the equation of mass proportional to $\eta_{\alpha\beta} \overset{2}{k}{}^{\alpha} \overset{2}{k}{}^{\beta}$. Again, if we use the equations of motion, this is $(\dot{v}, \dot{v})$, to the fourth approximation.

When we calculated the surface integrals, and so the equations of motion and of spin, we took the limit as the radius of the surface, S, tended to zero. This would be easily justified if the solution (1.11) existed in the limit, but it does not since the source function is of the order of γ^{-4} near the world line. What we must do is expand the source function in powers of γ near the world line and then separate it into a part which goes to infinity faster than γ^{-2} and a part that does not, $\overset{*}{A}{}^{\mu\nu}$ and $\overset{'}{A}{}^{\mu\nu}$ respectively. As our integral of the first part we take

$$(3.14) \qquad (4\pi)^{-1} \int \frac{1}{2} \overset{*}{A}{}^{\mu\nu} \cdot D(r - r') \cdot \mathrm{d}s .$$

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[1] A. PAPAPETROU: *Proc. Roy. Soc.*, **209**, 248 (1951).

for fixed a and analytically continue it inside the surface, $S(a)$. From our calculations, we know that this does not contribute to the equations of motion. To integrate $\acute{A}^{\mu\nu}$, we replace $\overset{*}{A}{}^{\mu\nu}$ by $\acute{A}^{\mu\nu}$ in (3.14) and take the limit as a tends to zero. This limit exists and gives the equations of motion found in this paper.

* * *

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RIASSUNTO (*)

Nel presente lavoro abbiamo calcolato in seconda approssimazione le equazioni del moto e dello spin col metodo esposto dall'autore nel primo articolo di questa serie. I risultati sono del tutto analoghi a quelli derivati da PAPAPETROU [1] per una particella polo-dipolo. Tuttavia, seguendo i concetti di EINSTEIN, INFELD e HOFFMAN, li abbiamo derivati dal campo esterno alla particella mentre PAPAPETROU li aveva derivati da un tensore impulso-energia.

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One Particle Singularities of Green Functions in Quantum Field Theory (*).

W. ZIMMERMANN

Physics Department, University of California, Berkeley - Cal.

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Summary. — The singularities of Green functions arising from intermediate vacuum and one particle states are investigated.

1. – Introduction.

As is well known from perturbation theory, the Fourier transform

$$(1) \qquad \tau(p_1 \dots p_n) = \frac{1}{(2\pi)^{5n/2}} \cdot \int \mathrm{d}x_1 \dots \mathrm{d}x_n \exp\left[-i\,(p_1 x_1 + \dots + p_n x_n)\right](\Omega, TA(x_1) \dots A(x_n)\Omega),$$

of the vacuum τ-function contains singularities of the form $\delta(p)$, $\delta(p^2+m^2)$ and $P(1/(p^2+m^2))$ in any variable $p = \sum_{\nu=1}^{r} p_{i_\nu}$. The singularities $\delta(p)$ arise from intermediate vacuum states; the singularities of the form $\delta(p^2+m^2)$ and $P(1/(p^2+m^2))$ are caused by intermediate one particle states (including stable bound states) of the discrete mass eigenvalue m.

As has been pointed out by CHEW and LOW [1] these one particle singularities

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(1) G. CHEW and F. LOW: to be published.

seem to play a significant part in the scattering theory. In the present paper we investigate the general structure of the vacuum and the one particle singularities of the τ-functions, in the case that all variables p_i are real four vectors. The one particle singularities of the analytical continuation of (1) will not be discussed in general. However in the case of forward scattering the method will be applied to dispersion relations in order to determine the indefinite constant appearing in Bogoljubov's treatment (2) of the one particle singularities. The result thus obtained is in agreement to Symanzik's form (3) of the one particle term in the dispersion relation for forward scattering.

2. – General conditions.

We consider the model of a neutral scalar field described by a hermitian operator $A(x)$ which is supposed to be causal, translation and Lorentz invariant. We further assume that no negative eigenvalues appear in the energy and rest mass spectrum.

In order to determine the one particle singularities of the τ-functions it is convenient to use the generalized definition of asymptotic fields which was recently given in connection with the bound state problem (4,5). If m^2 denotes a discrete eigenvalue of $-P_\mu^2$ the asymptotic field operators $B^m_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})$ are defined by

$$(2)\qquad B^m_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1}) = B(x;\,\xi_1 \dots \xi_{r-1}) + \int \mathrm{d}x' \Delta_{\substack{\text{Adv}\\\text{Ret}}}(m, x-x') K^m_{x'} B(x';\,\xi_1 \dots \xi_{r-1}),$$

with $r \geqslant 1$ and

$$(3)\qquad \begin{cases} B(x);\,\xi_1 \dots \xi_{r-1}) = TA(x_1) \dots A(x_r), \quad K^m_x = \Box_x - m^2 \\ x = \dfrac{1}{r}(x_1 + \dots + x_r), \\ \xi_i = x_i - x_{i+1};\ i = 1 \dots r-1. \end{cases}$$

The defined operators satisfy the Klein-Gordon equation

$$(4)\qquad (\Box_x - m^2) B^m_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1}) = 0$$

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(2) N. N. Bogoljubov, B. V. Medvedev and M. K. Polivanov: lecture notes. Translated at the Institute for Advanced Study (Princeton 1957).

(3) K. Symanzik: *Phys. Rev.*, **743**, 743 (1957).

(4) R. Haag: *Phys. Rev.*, **112**, 669 (1958); W. Zimmermann: *Nuovo Cimento*, **10**, 567 (1958).

(5) A related formulation which does not use asymptotic fields was given by K. Nishijima: *Progr. Theor. Phys.*, **111**, 995 (1958).

and are Lorentz and translation invariant

$$\frac{\partial}{\partial x^\mu} B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1}) = -i[P_\mu; B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1})], \tag{5}$$

$$B^m_{\substack{\text{out}\\\text{in}}}(Lx; L\xi_1 \dots L\xi_{r-1}) = U(L) B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1}) U(L)^{-1}. \tag{6}$$

Definition (2) is equivalent to the asymptotic relations [6,7]

$$\lim_{\substack{t\to\pm\infty\\ x_0=t}} \int \mathrm{d}_3x\, B(x; \xi_1 \dots \xi_{r-1}) \overleftrightarrow{\frac{\partial}{\partial x_0}} f^*(x) = \int \mathrm{d}_3x\, B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1}) \overleftrightarrow{\frac{\partial}{\partial x_0}} f^*(x), \tag{7}$$

if $f(x)$ is an arbitrary normalizable solution of $(\Box - m^2)f(x) = 0$. It can further be shown [4] that the operators $B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1})$ satisfy the commutation relations

$$[B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1}) B^m_{\substack{\text{out}\\\text{in}}}(y; \eta_1 \dots \eta_{s-1})] = \\ = (\Omega, [B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots \xi_{r-1}) B^m_{\substack{\text{out}\\\text{in}}}(y; \eta_1 \dots \eta_{s-1})] \Omega). \tag{8}$$

For $r = 1$, equ. (2) and (7) become the definition of the conventional asymptotic fields operators

$$A^m_{\substack{\text{out}\\\text{in}}}(x) = A(x) + \int \mathrm{d}x' \Delta_{\substack{\text{Adv}\\\text{Ret}}}(m, x - x') K^m_{x'} A(x'), \tag{2'}$$

$$\lim_{\substack{t\to\pm\infty\\ x_0=t}} \int \mathrm{d}_3x\, A(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f^*(x) = \int \mathrm{d}_3x\, A^m_{\substack{\text{out}\\\text{in}}}(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f^*(x). \tag{7'}$$

To investigate the vacuum singularities of the τ-functions it is useful to

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[6] The right hand side is independent of x since $B^m_{\substack{\text{out}\\\text{in}}}$ and f are solutions of the same Klein-Gordon equation.

[7] This relation is to be understood in the sense of weak operator convergence, *i.e.*, for any two normalizable states Φ, Ψ and any suitable test function $c(\xi_1 \dots \xi_{r-1})$ should hold

$$\lim_{\substack{t\to\pm\infty\\ x_0=t}} \int \mathrm{d}_3x\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{r-1} c(\xi_1 \dots \xi_{r-1}) (\Phi, B(x; \xi_1 \dots \xi_{r-1})\psi) \overleftrightarrow{\frac{\partial}{\partial x^0}} f^*(x) = \\ = \int \mathrm{d}_3x\, \mathrm{d}\xi_1 \dots d\xi_{r-1} c(\xi_1 \dots \xi_{r-1}) (\Phi, B^m_{\substack{\text{out}\\\text{in}}}(x; \xi_1 \dots {}^{+}{}_{r-1})\psi) \overleftrightarrow{\frac{\partial}{\partial x^0}} f^*(x).$$

if $f(x)$ is a solution of $(\Box - m^2)f(x) = 0$.

introduce the following asymptotic operators:

$$(9)\quad \begin{cases} B^0_{\text{in}}(x;\,\xi_1 \dots \xi_{r-1}) = B(x;\,\xi_1 \dots \xi_{r-1}) - \int \mathrm{d}x'^0 \theta(x^0 - x'^0)\, \dfrac{\partial}{\partial x'^0}\, B(x'^0, \boldsymbol{x};\,\xi_1 \dots \xi_{r-1}), \\ B^0_{\text{out}}(x;\,\xi_1 \dots \xi_{r-1}) = B(x;\,\xi_1 \dots \xi_{r-1}) + \int \mathrm{d}x'^0\, \theta(x'^0 - x^0)\, \dfrac{\partial}{\partial x'^0}\, B(x'^0, \boldsymbol{x};\,\xi_1 \dots \xi_{r-1}). \end{cases}$$

These operators are Lorentz and translation invariant as can be shown by transforming (9) in momentum space:

$$(10)\qquad \frac{\partial}{\partial x^\mu}\, B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1}) = -\,i[P_\mu,\, B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})]\,,$$

$$(11)\qquad B^0_{\substack{\text{out}\\\text{in}}}(Lx;\, L\xi_1 \dots L\xi_{r-1}) = U(L)\, B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})\, U(L)^{-1}\,.$$

Differentiation of (9) with respect to x_0 gives

$$\frac{\partial B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})}{\partial x^0} = 0\,.$$

Since this holds for any value of $\xi_1 \dots \xi_{r-1}$ we have also

$$\frac{\partial}{\partial x^i}\, B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1}) = 0\,, \qquad i = 1,\ 2,\ 3$$

because of (11). Hence $B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})$ does not depend on x and will therefore be denoted by

$$(12)\qquad B^0_{\substack{\text{out}\\\text{in}}}(\xi_1 \dots \xi_{r-1}) = B^0_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})\,.$$

3. – Vacuum singularities.

The general structure of the vacuum singularities of the τ-function (1) was first given by WATANABE in the form ([8])

$$(13)\qquad \tau(p_1 \dots p_n) = \delta(p_1 + \dots + p_n)\, \tilde{\eta}(p_1 \dots p_n) + \\ + \sum_A \delta(p_{i_1} + p_{i_2} + \dots)\, \delta(p_{j_1} + p_{j_2} + \dots) \dots \tilde{\eta}(p_{i_1} p_{i_2} \dots)\, \tilde{\eta}(p_{j_1} p_{j_2} \dots) \dots\,.$$

([8]) I. WATANABE: *Prog. Theor. Phys.*, **4**, 371 (1953); K. SYMANZIK: *Zeits. f. Naturf.*, **10***a*, 809 (1954).

(The sum $\sum_A$ is taken over all possible partitions A of the indices $1, \ldots, n$ in distinct classes $i_1 i_2 \ldots ; j_1 j_2 \ldots ; \ldots$) where $\tilde{\eta}(p_1 \ldots p_k)$ does not contain any singularities of the form $\delta(\sum_{\nu=1}^{r} p_{i_\nu})$.

In this section we will derive Watanabe's result without using perturbation theory. For this aim we have only to show that the function $\tilde{\eta}(p_1 \ldots p_n)$ defined recursively by [8,9]

$$(14)\quad \begin{cases} (\Omega, TA(x_1) \ldots A(x_n)\Omega) = \eta(x_1 \ldots x_n) + \sum_A \eta(x_{i_1} x_{i_2} \ldots)\eta(x_{j_1} x_{j_2} \ldots) \ldots \\ \delta(\sum_{\nu=1}^{n} p_\nu)\tilde{\eta}(p_1 \ldots p_n) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp\left[-i(p_1 x_1 + \ldots + p_n x_n)\eta(x_1 \ldots x_n)\right]. \end{cases}$$

is free from any vacuum singularities $\delta(\sum_{\nu=1}^{r} p_{i_\nu})$.

Using the operators $B^0_{\substack{\mathrm{out}\\ \mathrm{in}}}(\xi_1 \ldots \xi_{r-1})$ we obtain the following relations from (9)

$$(15)\quad T(A(x_{r+1}) \ldots A(x_n))\, B^0_{\mathrm{in}}(\xi_1 \ldots \xi_{r-1}) = \\ = TA(x_1) \ldots A(x_n) - \int \mathrm{d}x'^0 \theta(x^0 - x'^0) \frac{\partial}{\partial x'^0} TA(x'_1) \ldots A(x'_r) A(x_{r+1}) \ldots A(x_n),$$

$$(15')\quad \begin{cases} B^0_{\mathrm{out}}(\xi_1 \ldots \xi_{r-1}) TA(x_{r+1}) \ldots A(x_n) = \\ = TA(x_1) \ldots A(x_n) + \int \mathrm{d}x'^0 \theta(x'^0 - x^0) \frac{\partial}{\partial x'^0} TA(x'_1) \ldots A(x'_r) A(x_{r+1}) \ldots A(x_n), \\ x'^i_\nu = x^i_\nu,\ x'^0_\nu - x'^0_{\nu+1} = x^0_\nu - x^0_{\nu+1}, \qquad x'^0 = \frac{1}{r}\sum_{\nu=1}^{r} x'^0_\nu, \qquad x^0 = \frac{1}{r}\sum_{\nu=1}^{r} x^0_\nu. \end{cases}$$

Taking the sum of the vacuum expectation values of (15), (15′) we get

$$(16)\quad (\Omega, TA(x_1) \ldots A(x_n)\Omega) - \\ - \tfrac{1}{2}\int \mathrm{d}x'^0 \varepsilon(x^0 - x'^0) \frac{\partial}{\partial x'^0} (\Omega, TA(x'_1) \ldots A(x'_r) A(x_{r+1}) \ldots A(x_n)\Omega) = \\ = \tfrac{1}{2}(\Omega, B^0_{\mathrm{out}}(\xi_1 \ldots \xi_{r+1}) TA(x_{r-1}) \ldots A(x_n)\Omega) + \\ + \tfrac{1}{2}(\Omega, TA(x_{r+1}) \ldots A(x_n) B^0_{\mathrm{in}}(\xi_1 - \xi_{r-1})\Omega) = \\ = (\Omega, TA(x_1) \ldots A(x_r)\Omega)(\Omega, TA(x_{r+1}) \ldots A(x_n)\Omega),$$

where we have used

$$(\Omega, B^0_{\substack{\mathrm{out}\\ \mathrm{in}}}(\xi_1 \ldots \xi_{r-1})\Phi) = 0 \qquad \text{for } (\Omega, \Phi) = 0.$$

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[9] These functions have also been used in many other papers, for instance P. KRISTENSEN: *Dan. Mat. Fys. Medd.*, **28**, 12 (1954); E. FREESE: *Nuovo Cimento*, **2**, 50 (1955).

(because of translation invariance (10)) and

$$(17)\qquad (\Omega, B^0_{\text{in}}(\xi_1 \dots \xi_{r-1})\Omega) = (\Omega, B(x; \xi_1 \dots \xi_{r-1})\Omega) = (\Omega, B^0_{\text{out}}(\xi_1 \dots \xi_{r-1})\Omega)\,.$$

Relation (17) follows from (9) and

$$\frac{\partial}{\partial x^0}(\Omega, B(x; \xi_1 \dots \xi_{r-1})\Omega) = -i(\Omega, [P_0, B(x; \xi_1 \dots \xi_{r-1})]\Omega) = 0\,.$$

In momentum space relation (16) becomes

$$\frac{1}{\sum_{\nu=1}^{r} p^0_\nu}\Big\{\Big(\sum_{\nu=1}^{r} p^0_\nu\Big)\tau(p_1 \dots p_n)\Big\} = \tau(p_1 \dots p_r)\,\tau(p_{r+1}\, p_n)\,.$$

Since $\tau(p_1 \dots p_n)$ is a symmetric function this relation can be written in the more general form

$$(18)\qquad \frac{1}{\sum_{\nu=1}^{r} p^0_{i_\nu}}\Big\{\Big(\sum_{\nu=1}^{r} p^0_{i_\nu}\Big)\tau(p_1 \dots p_n)\Big\} = \tau(p_1 \dots p_n) - \tau(p_{i_1} \dots p_{i_r})\,\tau(p_{i_{r+1}} \dots p_{i_n})\,.$$

Now we expand $\tilde{\eta}(p_1 \dots p_n)$ with respect to δ-functions $\delta\big(\sum_{\nu=1}^{r} p_{i_\nu}\big)$

$$(19)\qquad \delta\Big(\sum_{\nu=1}^{n} p_\nu\Big)\,\tilde{\eta}(p_1 \dots p_n) = \delta\Big(\sum_{\nu=1}^{n} p_\nu\Big)\, g(p_1 \dots p_n) + \sum_{A} \delta(p_{i_1} + p_{i_2} + \dots)\,\delta(p_{j_1} + p_{j_2} + \dots) \dots g_A(p_1 \dots p_n)\,,$$

and try to determine the coefficients g, g_A. By complete induction we shall show that

$$(20)\qquad \tilde{\eta}(p_1 \dots p_n) = g(p_1 \dots p_n)\,.$$

We assume that (20) has already been shown for $1, \dots, n-1$. Then, inserting (19) into (18) we get

$$(18')\qquad \begin{cases} \dfrac{1}{\sum_{\nu=1}^{r} p^0_{i_\nu}}\Big\{\Big(\sum_{\nu=1}^{r} p^0_{i_\nu}\Big)\tilde{\eta}(p_1 \dots p_n)\Big\} = \tilde{\eta}(p_1 \dots p_n)\,, \\ i_\nu = 1, \dots, n-1\,, \end{cases}$$

for any partial sum $q = \sum_{\nu=1}^{r} p_{i_\nu}$. Repeated application of (18′) leads to

$$\tilde{\eta}(p_1 \dots p_n) = \prod_{\mu=1}^{m} \frac{1}{q^0_\mu}\Big\{\prod_{\mu=1}^{m} q^0_\mu\, \tilde{\eta}(p_1 \dots p_n)\Big\} = g(p_1 \dots p_n)\,,$$

where $q_1 \dots q_m$ denote all partial sums $\sum_{\nu=1}^{r} p_{i_\nu}$ which can be formed from the coordinates $p_1 \dots p_{n-1}$. So we have shown that (13) gives in fact the correct expansion of $\tau(p_1 \dots p_n)$ with respect to the δ-function $\delta(\sum_{\nu=1}^{r} p_{i_\nu})$.

4. – One particle singularities.

In this section we investigate the one particle singularities of the τ-functions. Let $m_1^2, \dots, m_l^2$ denote the discrete non-vanishing eigenvalues of $-P_\mu^2$. We shall show that the function $\tilde{\eta}(p_1 \dots p_n)$ can be expressed by

$$\tilde{\eta}(p_1 \dots p_n) = \prod_{\alpha=1}^{l} \prod_{\mu=1}^{m} \lim_{\varepsilon \to +0} \frac{1}{q_\mu^2 + m_\alpha^2 - i\varepsilon}\, \sigma(p_1 \dots p_n)\,, \tag{21}$$

$$\sum_{\nu=1}^{n} p_\nu = 0\,,$$

where $q_1 \dots q_m$ denote all partial sums $\sum_{\nu=1}^{r} p_{i_\nu}$ which can be formed from the co-ordinates $p_1 \dots p_{n-1}$. The function

$$\sigma(p_1 \dots p_n) = \prod_{\alpha=1}^{l} \prod_{\mu=1}^{m} (q_\mu^2 + m_\alpha^2)\, \tilde{\eta}(p_1 \dots p_n)\,, \tag{21'}$$

contains no vacuum or one particle singularities any more.

We start investigating the singularities $1/(p^2+m^2)$, $\delta(p^2+m^2)$ where p denotes one of the co-ordinates p_i of $\tau(p_1 \dots p_n)$ and m^2 denotes a discrete eigenvalues of $-P_\mu^2$.

From relation (7′) it follows that

$$\lim_{t \to +\infty} \int_{x_0 = t} \mathrm{d}_3 x_1\, f(x_1) \overleftrightarrow{\frac{\partial}{\partial x_1^0}}\, TA(x_1) \dots A(x_n) = \int \mathrm{d}_3 x_1\, f(x_1) \overleftrightarrow{\frac{\partial}{\partial x_1^0}}\, A_{\mathrm{out}}^m(x_1)\, TA(x_2) \dots A(x_n)\,,$$

$$\lim_{t \to -\infty} \int_{x_0 = t} \mathrm{d}_3 x_1\, f(x_1) \overleftrightarrow{\frac{\partial}{\partial x_1^0}}\, TA(x_1) \dots A(x_n) = \int \mathrm{d}_3 x_1\, f(x_1) \overleftrightarrow{\frac{\partial}{\partial x_1^0}}\, T(A(x_2) \dots A(x_n))\, A_{\mathrm{in}}^m(x_1)\,,$$

for any normalizable solution of $(\Box - m^2)\, f(x) = 0$. This may be written in the equivalent form

$$T(A(x_2) \dots A(x_n))\, A_{\mathrm{in}}^m(x_1) = \tag{22}$$

$$= TA(x_1) \dots A(x_n) + \int \mathrm{d}x_1'\, \Delta_{\mathrm{Ret}}(m_1, x_1 - x_1')\, K_{x_1'}^m\, TA(x_1')\, A(x_2) \dots A(x_n)\,,$$

$$(22') \quad A^m_{\text{out}}(x_1)\, TA(x_1) \dots A(x_n) = \\ = TA(x_1) + \int \mathrm{d}x_1' \Delta_{\text{Adv}}(m_1, x_1 - x_1') K^m_{x_1} TA(x_1') A(x_2) \dots A(x_n) .$$

Subtracting (22) from (22′) we obtain the reduction formula

$$(23) \quad A^m_{\text{out}}(x_1) TA(x_2) \dots A(x_n) - T(A(x_2) \dots A(x_n)) A^m_{\text{in}}(x_1) = \\ = \int \mathrm{d}x_1' \Delta(m_1, x_1 - x_1') K^m_{x_1} TA(x_1') A(x_2) \dots A(x_n) .$$

As a consequence of translation invariance (6) and the positive definiteness of the energy spectrum we have

$$\left.\begin{aligned} \int \mathrm{d}x \exp[-ipx]\, A^m_{\text{in}}(x)\Omega = 0 \\ \int \mathrm{d}x \exp[-ipx]\, A^m_{\text{out}}(x)\Omega = 0 \end{aligned}\right\} \quad \text{for } p_0 > 0 .$$

The vacuum expectation value of (23) may therefore be split into the positive and the negative frequency parts

$$(24) \quad (\Omega, A^m_{\text{out}}(x_1) TA(x_2) \dots A(x_n)\Omega) = \\ = \int \mathrm{d}x_1' \Delta^+(m, x_1 - x_1') K^m_{x_1'}(\Omega, TA(x_1') A(x_2) \dots A(x_n)\Omega) .$$

$$(24') \quad (\Omega, T(A(x_2) \dots A(x_n)) A^m_{\text{in}}(x_1)\Omega) = \\ = -\int \mathrm{d}x_1' \Delta^-(m, x_1 - x_1') K^m_{x_1'}(\Omega, TA(x_1') A(x_2) \dots A(x_n)\Omega) .$$

Inserting (24) into (22′) we get

$$(25) \quad (\Omega, TA(x_1) \dots A(x_n)\Omega) = \int \mathrm{d}x_1' \Delta_F(m, x_1 - x_1') K^m_{x_1'}(\Omega, TA(x_1') A(x_2) - A(x_n)\Omega) ,$$

or in momentum space

$$\tau(p_1 \dots p_n) = \lim_{\varepsilon \to +0} \frac{1}{p_i^2 + m^2 - i\varepsilon} \{(p_i^2 + m^2)\tau(p_i \dots p_n)\} .$$

In terms of η-functions (defined recursively by equ. (14)) relation (25) takes the form

$$(25') \quad \eta(x_1 \dots x_n) = \int \mathrm{d}x_1' \Delta_F(m, x_1 - x_1') K^m_{x_1'} \eta(x_1' x_2 \dots x_n) .$$

Applying (25′) successively to the co-ordinates $x_1 \dots x_n$ we get finally

$$(26)\quad \eta(x_1 \dots x_n) = \int \mathrm{d}x_1' \dots \mathrm{d}x_n' \Delta_F(m, x_1 - x_1') \dots \Delta_F(m, x_n - x_n') K_{x_1'}^m \dots K_{x_n'}^m \eta(x_1' \dots x_n') .$$

This formula is well known from perturbation theory.

Now we go one step further and use the generalized asymptotic fields (2) for an arbitrary number $r \geqslant 1$ of co-ordinates. As a generalization of (22), (22′) we obtain

$$(27)\quad T(A(x_{r+1}) \dots A(x_n)) B_{\mathrm{in}}^m(x; \xi_1 \dots \xi_{r-1}) = TA(x_1) \dots A(x_n) + \\ + \int \mathrm{d}x' \Delta_{\mathrm{Ret}}(m, x - x') K_{x'}^m TA(x_1') - A(x_r') A_{r+1}) \dots A(x_n) ,$$

$$(27')\quad B_{\mathrm{out}}^m(x; \xi_1 \dots \xi_{r-1}) TA(x_{r+1}) \dots A(x_n) = \\ = TA(x_1) \dots A(x_n) + \int \mathrm{d}x' \Delta_{\mathrm{Adv}}(m, x - x') K_{x'}^m TA(x_1') \dots A(x_r') A(x_{r+1}) \dots A(x_n) ,$$

with

$$x = \frac{1}{r} \sum_{\nu=1}^{r} x_\nu, \quad x' = \frac{1}{r} \sum_{\nu=1}^{r} x_\nu' , \quad \xi_i = x_i - x_{i+1} = x_i' - x_{i+1}' ,$$

$$i = 1, \dots, r-1 .$$

Taking the difference of (27) and (27′) we get the reduction formula

$$(28)\quad B_{\mathrm{out}}^m(x; \xi_1 \dots \xi_{r-1}) TA(x_{r+1}) \dots A(x_n) - T(A(x_{r+1}) \dots A(x_n)) \cdot B_{\mathrm{in}}^m(x, ; \xi_1 \dots \xi_{r-1}) = \\ = \int \mathrm{d}x' \Delta(m, x - x') K_{x'}^m TA(x_1') \dots A(x_r') A(x_{r+1}) \dots A(x_n) .$$

Therefore

$$(29)\quad (\Omega, B_{\mathrm{out}}^m(x; \xi_1 \dots \xi_{r-1}) TA(x_{r+1}) \dots A(x_n)\Omega) = \\ = \int \mathrm{d}x' \Delta^+(m, x - x') K_{x'}^m (\Omega, TA(x_1') \dots A(x_r') A(x_{r+1}) \dots A(x_n)\Omega) .$$

$$(29')\quad (\Omega, T(A(x_{r+1}) \dots A(x_n)) B_{\mathrm{in}}^m(x; \xi_1 \dots \xi_{r-1})\Omega) = \\ - \int \mathrm{d}x' \Delta^-(m, x - x') K_{x'}^m (\Omega, TA(x_1') \dots A(x_r') A(x_{r+1}) \dots A(x_n)\Omega) .$$

Inserting (29) into (27′) we obtain

$$(30)\quad (\Omega, TA(x_i)_1 \dots A(x_n)\Omega) = \\ = \int \mathrm{d}x' \Delta_F(m, x - x') K_{x'}^m (\Omega, TA(x_1') \dots A(x_r') A(x_{r+1}) \dots A(x_n)\Omega) ,$$

or in momentum space

$$(30') \qquad \tau(p_1 \dots p_n) = \lim_{\varepsilon \to +0} \frac{1}{(\sum_{\nu=1}^{r} p_{i_\nu})^2 + m^2 - i\varepsilon} \left((\sum_{\nu=1}^{r} p_{i_\nu})^2 + m^2 \right) \tau(p_1 \dots p_n) ,$$

and

$$(30'') \qquad \tilde{\eta}(p_1 \dots p_n) = \lim_{\varepsilon \to +0} \frac{1}{(\sum_{\nu=1}^{r} p_{i_\nu})^2 + m^2 - i\varepsilon} \left((\sum_{\nu=1}^{r} p_{i_\nu})^2 + m^2 \right) \tilde{\eta}(p_1 \dots p_n) .$$

Repeated application of this formula with respect to any mass m_α and to any partial sum q_μ of the variables p_i $(i = 1, \dots, n-1)$ leads finally to

$$\tilde{\eta}(p_1 \dots p_n) = \prod_{\alpha=1}^{l} \prod_{\mu=1}^{m} \frac{1}{q_\alpha^2 + m_\mu^2 - i\varepsilon} \sigma(p_1 \dots p_n) ,$$

$$\sum_{\nu=1}^{n} p_\nu = 0 ,$$

where

$$(32) \qquad \sigma(p_1 \dots p_n) = \prod_{\alpha=1}^{l} \prod_{\mu=1}^{m} (q_\mu^2 + m_\alpha^2) \tilde{\eta}(p_1 \dots p_n) .$$

The function $\sigma(p_1 \dots p_n)$ contains no vacuum or one particle singularities any more. If one of the variables $q = \sum_{\nu=1}^{r} p_{i_\nu}$ lies on the energy shell $p^2 + m_\alpha^2 = 0$ the function $\sigma(p_1 \dots p_n)$ can be expressed by a product of η-functions. In order to show this we calculate the expression

$$\left((\sum_{\nu=1}^{r} p_\nu)^2 + m^2 \right) \tau(p_1 \dots p_n) =$$

$$= \int \mathrm{d}x_1 \dots \mathrm{d}x_n \exp\left[-i \sum_{\nu=1}^{n} p_r x_r \right] K_x^m(\Omega, TA(x_1) \dots A(x_n)\Omega) ,$$

for

$$(\sum_{\nu=1}^{r} p_\nu)^2 + m^2 = 0 , \quad x = \frac{1}{r} \sum_{\nu=1}^{r} x_\nu .$$

Taking the sum of (29), (29′) we obtain the relation

$$\delta(p^2 + m^2) \int \mathrm{d}x \exp[-ipx] K_x^m(\Omega, TA(x_1) \dots A(x_n)\Omega) =$$

$$= \frac{i}{2\pi} \int \mathrm{d}x \exp[-ipx] \left\{ (\Omega, B_{\mathrm{out}}^m(x; \xi_1 \dots \xi_{r-1}) TA(x_{r+1}) \dots A(x_n)\Omega) + \right.$$

$$\left. + (\Omega, TA((x_{r+1}) \dots A(x_n)(B_{\mathrm{in}}^m(x; \xi_1 \dots \xi_{r-1})\Omega) \right\} ,$$

$$= \frac{i}{2\pi} \int \mathrm{d}x \exp[-ipx] \int \frac{\mathrm{d}_3 k}{2\sqrt{\boldsymbol{k}^2 + m^2}} \left\{ (\Omega, TA(x_1) \dots A(x_{nr}) \Phi_k^m)(\Phi_k^m, TA(x_{r+1}) \dots A(x_n)\Omega \right.$$

$$\left. + (\Omega, TA(x_{r+1}) \dots A(x_n)\Phi_k^m)(\Phi_k^m, TA(x_1) \dots A(x_r)\Omega) \right\} ,$$

where Φ_k^m denotes the eigenstate of P_μ with momentum k_μ and mass m, normalized by

$$(\Phi_k^m, \Phi_{k'}^m) = 2\,|\sqrt[0]{\boldsymbol{k}^2 + m^2}\,|\,\delta_3(\boldsymbol{k} - \boldsymbol{k}')\,.$$

In the last line of (33) we have used

$$(\Omega, B^m_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})\Phi) = 0 \quad \text{if} \quad -P_\mu^2\Phi = \varkappa^2\Phi \quad \text{and} \quad \varkappa^2 \neq m^2$$

and

$$(\Omega, B^m_{\substack{\text{out}\\\text{in}}}(x;\,\xi_1 \dots \xi_{r-1})\Phi_k^m) = (\Omega, B(x;\,\xi_1 \dots \xi_{r-1})\Phi_k^m) = (\Omega, TA(x_1) \dots A(x_r)\Phi_k^m)\,,$$

which follows from (2) and translation invariance (5). We rearrange equ. (33) with the help of the reduction formula ([10])

$$(34)\qquad (\Omega, TA(x_1) \dots A(x_l)\Phi_k^m) = -\frac{i}{(2\pi)^{\frac{3}{2}} c_m^{\frac{1}{2}}} \cdot \int \mathrm{d}z\, \exp[ikz]\, K_z^m(\Omega, TA(x_1) \dots A(x_l)\,A(z)\Omega)\,.$$

The normalization constant c_m is defined by ([10])

$$c_m^{-\frac{1}{2}}(\Omega, A(z)\Phi_k^m) = \frac{\exp[ikz]}{(2\pi)^{\frac{3}{2}}}\,.$$

Inserting (34) and the corresponding reduction formula for $(\Phi_k^m, TA(x_1) \dots A(x_l)\Omega)$ into (33) we get

$$\delta(p^2 + m^2)\int \mathrm{d}x\, \exp[-ipx]\, K_x^m(\Omega, TA(x_1) \dots A(x_n)\Omega) = \\ = \frac{i}{2\pi c_m}\int \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}z'\, \exp[-ipx]\, K_z^m(\Omega, TA(x_1) \dots A(x_r)\,A(z)\Omega) \cdot \\ \cdot\, \Delta_1(z - z')\, K_{z'}^m(\Omega, TA(x_{r+1}) \dots A(x_n)\,A(z')\Omega)\,.$$

This leads finally to

$$\delta(p^2 + m^2)\,\{(p^2 + m^2)\,\tau(p_1 \dots p_n)\} = \\ = \frac{2\pi i}{c_m}\int \mathrm{d}q\,\{(q^2 + m^2)\tau(p_1 \dots p_r q)\}\,\delta(q^2 + m^2)\,\{(q^2 + m^2)\,\tau(p_{r+1} \dots p_n, -q)\}\,,$$

([10]) We assume here that

$$(\Omega, A(z)\Phi_k^m) \neq 0\,,$$

The case that $(\Omega, A(z)\Phi_k^m) \equiv 0$ but $(\Omega, A(z_1)A(z_2)\Phi_k^m) \neq 0$, will be discussed later.

or in terms of η-functions

$$(35)\qquad (p^2+m^2)\tilde{\eta}(p_1 \dots p_n) = \\ = \frac{2\pi i}{c_m}\{(p^2+m^2)\,\tilde{\eta}(p_1 \dots p_r p)\}\{(p^2+m^2)\,\tilde{\eta}(p_{r+1},\dots,p_n,-p)\}\,,$$

for

$$p^2+m^2=0\,,\qquad p=\sum_{\nu=1}^{r} p_\nu\,,\qquad \sum_{\nu=1}^{n} p_\nu = 0\,.$$

Since $\tilde{\eta}(p_1 \dots p_n)$ is symmetric the relation

$$(36)\qquad \begin{cases} (p^2+m^2)\tilde{\eta}(p_1 \dots p_n) = \\ \qquad = \dfrac{2\pi i}{c_m}\{(p^2+m^2)\tilde{\eta}(p_{i_1} \dots p_{i_r} p)\}\{(p^2+m^2)\tilde{\eta}(p_{i_{r+1}} \dots p_{i_n},-p)\}\,, \\ p=\sum_{\nu=1}^{r} p_{i_\nu}\,,\quad p^2+m^2=0 \end{cases}$$

holds for any variable $p=\sum_{\nu=1}^{r} p_{i_\nu}$ if $p^2+m^2=0$. In this way the function $\sigma(p_1 \dots p_n)$, defined by (32), may be expressed by the product (36) if the variable $p=\sum_{\nu=1}^{r} p_{i_\nu}$ lies on the energy shell $p^2+m^2=0$.

In case that the states Φ belonging to the mass m satisfy

$$(\Omega,\, A(x)\Phi) = 0\,,\qquad (\Omega,\, A(x)\,A(y)\Phi) \neq 0$$

a modified reduction formula ([11]) has to be used instead of (34).

The final result is in this case

$$(37)\quad \delta(p^2+m^2)\tau(p_1 \dots p_n) = -\frac{i}{2\pi}\int \mathrm{d}x_1 \dots \mathrm{d}x_n\,\mathrm{d}z\,\mathrm{d}z'\,K_z^m(\Omega,\, TA(x_1) \dots A(x_r)B(z)\Omega)\cdot \\ \cdot\, \Delta_1(z-z')\,K_{z'}^m(\Omega,\, TA(x_{r+1}) \dots A(x_n)B(z)\Omega)\,,$$

$$= 2\pi i\int \mathrm{d}q\,\{(q^2+m^2)\,\tau(p_1 \dots p_r\,|\,q)\}\,\delta(q^2+m^2)\,\{(q^2+m^2)\,\tau(p_{r+1} \dots p_n\,|\,-q)\}\,,$$

with

$$\tau(p_1 \dots p_k\,|\,q) = \frac{1}{(2\pi)^{5(k+1)/2}}\cdot \\ \cdot \int \mathrm{d}x_1 \dots \mathrm{d}x_k\,\mathrm{d}z\,\exp\left[i\Big(\sum_{\nu=1}^{k} p_\nu x_\nu + qz\Big)\right](\Omega,\, TA(x_1) \dots A(x_k)B(z)\Omega)\,,$$

provided that the limit [11]

$$B(z) = \lim_{\zeta \to 0} \frac{TA(z+\zeta)A(z-\zeta) - (\Omega, TA(\zeta)A(-\zeta)\Omega)}{(2\pi)^{\frac{3}{2}}(\Omega, TA(\zeta)A(-\zeta)\Phi_0)},$$

exists. In terms of η-functions equ. (37) becomes

$$(38) \quad \begin{cases} (p^2 + m^2)\,\tilde{\eta}(p_1 \dots p_n) = 2\pi i\,\{(p^2 + m^2)\,\tilde{\eta}(p_{i_1} \dots p_{i_r} | p)\} \cdot \\ \qquad\qquad \cdot \{(p^2 + m^2)\,\tilde{\eta}(p_{i_{r+1}} \dots p_{i_n} | -p)\} \\ p^2 + m^2 = 0\,, \qquad p = \sum_{\nu=1}^{r} p_{i_\nu}\,, \qquad \sum_{\nu=1}^{n} p_\nu = 0\,. \end{cases}$$

5. – Application to dispersion relations (forward scattering).

In this Section we consider a charged « nucleon » field $\psi(x)$ of mass M and spin zero interacting with a neutral meson field $A(x)$ of mass m ($m \leqslant M$) and spin zero. We assume that no other particles or bound states are present. The amplitude $f(p, k)$ for meson-nucleon forward scattering is given by [12]

$$(39) \quad \begin{cases} f(p, k) = -\int \mathrm{d}\xi \exp[-ik\xi]\,(\Box_\xi - M^2)^2 \cdot \\ \qquad \cdot (\Phi_\nu, [T\psi(\xi/2)\bar{\psi}(-\xi/2) - \langle T\psi(\xi/2)\,\bar{\psi}(-\xi/2)\rangle_0]\Phi_\nu) \\ p^2 = -m^2\,, \quad k^2 = -M^2\,, \end{cases}$$

p and k denote the momenta of the incoming (or outgoing) meson and nucleon. We use the Lorentz system in which the meson is at rest

$$\boldsymbol{p} = 0\,, \qquad p_0 = m\,.$$

In the proof of the dispersion relations [13-18] for forward scattering usually

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(11) W. Zimmermann: *Nuovo Cimento*, **10**, 567 (1958).

(12) $\langle\ \rangle_0$ denotes the vacuum expectation value.

(13) M. L. Goldberger: *Phys. Rev.*, **99**, 979 (1955).

(14) R. Oehme: *Nuovo Cimento*, **10**, 1316, (1956).

(15) N. N. Bogoljubov, B. V. Medvedev and M. K. Polivanov: lecture notes. Translated at the Institute for Advanced Study (Princeton 1957).

(16) R. Jost and H. Lehmann: *Nuovo Cimento*, **5**, 1598 (1957).

(17) K. Symanzik: *Phys. Rev.*, **105**, 743 (1957).

(18) H. J. Bremermann, R. Oehme and J. G. Taylor: *Phys. Rev.*, **109**, 2178 (1958).

the retarded function

$$
(40)\qquad \begin{cases} f_{\mathrm{Ret}}(p,k) = -\int \mathrm{d}\xi \exp[-ik\xi](\Box_\xi - M^2)^2 \cdot \\ \qquad \cdot (\Phi_\lambda, [R\psi(\xi/2)\,\overline{\psi}(-\xi/2) - \langle R\psi(\xi/2)\,\overline{\psi}(-\xi/2)\rangle_0]\,\Phi_\nu)\,, \\ R\psi(x)\,\overline{\psi}(y) = -i\vartheta(x-y)[\psi(x)\,\overline{\psi}(y)]\,, \end{cases}
$$

is used which is related to (39) by

$$
(41)\qquad \begin{cases} f_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = -i f(k_0, \boldsymbol{k}^2) \qquad \text{for } k_0 > 0\,, \\ f_{\mathrm{Rer}}(-k_0, \boldsymbol{k}^2) = f_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)^*\,. \end{cases}
$$

1) As is well known, the singularities of $f_{\mathrm{Ret}}(p,k)$ at $(k \pm p)^2 + M^2 = 0$ arising from intermediate one nucleon states cause additional complication in the proof of dispersion relations, since they contribute to the imaginary part of $f_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)$ in the domain $|k_0| < M$. A very simple treatment of this difficulty was given by BOGOLJUBOV (15). Following BOGOLJUBOV's proposal we remove the one particle singularities of f_{Ret} by forming

$$
(42)\qquad f'_{\mathrm{Ret}}(p,k) = ((k+p)^2 + M^2)((k-p)^2 + M^2) f_{\mathrm{Ret}}(p,k)\,.
$$

For $|k_0| < M$ holds

$$
\mathrm{Im}\, f'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0\,, \qquad |k_0| < M.
$$

Hence it can be shown by standard methods (14-18) that

$$
f'_{\mathrm{Ret}}(\omega, \omega^2 - M^2)
$$

has an analytical continuation which is regular in the upper half plane.

In order to get a dispersion relation for $f_{\mathrm{Ret}}(p,k)$ itself we will now use the results of the last section. The scattering amplitude (39) is related to the corresponding η-function by

$$
(43)\qquad f(p,k) = (2\pi)^3 (p^2 + m^2)^2 (k^2 + M^2)^2 \tilde{\eta}(-p, p; -k, k)\,.
$$

According to (14) $\tilde{\eta}(p, p'; k, k')$ is defined by

$$
\delta(p + p' + k + k')\tilde{\eta}(p, p'; k, k') = \tau(p, p'; k, k') - \tau_N(k, k')\tau_m(p, p')\,,
$$

$$
\tau(p, p'; k, k') = \frac{1}{(2\pi)^{10}} \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y' \exp[-i(py + p'y' + kx + k'x')] \cdot \\ \cdot (\Omega, T\psi(x)\,\overline{\psi}(x')A(y)A(y')\Omega)\,,
$$

$$\tau_N(k, k') = \frac{1}{(2\pi)^5}\int \mathrm{d}x\,\mathrm{d}x' \exp[-i(kx + k'x')](\Omega, T\psi(x)\,\psi(x')\Omega)\,,$$

$$\tau\ (p, p') = \frac{1}{(2\pi)^5}\int \mathrm{d}y\,\mathrm{d}y' \exp[-i(py + p'y')](\Omega, TA(y)\,A(y')\Omega)\,.$$

The one particle singularities of $\tilde{\eta}$ at $(k+p)^2 + M^2 = 0$, $(k+p')^2 + M^2 = 0$ are given by (compare equ. (31))

(44) $$(k^2 + M^2)(k'^2 + M^2)(p^2 + m^2)(p'^2 + m^2)\,\tilde{\eta}(p, p'; k, k') =$$

$$= \frac{1}{(k+p)^2 + M^2 - i\varepsilon}\,\frac{1}{(k+p')^2 + M^2 - i\varepsilon}\,\sigma(p, p'; k, k'), \qquad \varepsilon \to +0\,.$$

with

(44′) $$\sigma(p, p'; k, k') = (k^2 + M^2)(k'^2 + M^2)(p^2 + M^2)(p'^2 + m^2)\cdot$$

$$\cdot((k+p)^2 + M^2)((k+p')^2 + M^2)\,\tilde{\eta}(p, p'; k, k')\,.$$

For forward scattering (44) becomes

(45) $$f(p, k) = \frac{1}{(k+p)^2 + M^2 - i\varepsilon}\,\frac{1}{(k-p)^2 + M^2 - i\varepsilon}\,f'(p, k)\,, \qquad \varepsilon \to +0$$

with

(45′) $$f'(p, k) = ((k+p)^2 + M^2)((k-p)^2 + M^2)\,f(p, k)\,,$$

$$= (2\pi)^3\sigma(-p, p; -k, k)\,.$$

Using (41) and

$$f'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = -if'(k_0, \boldsymbol{k}^2) \qquad \text{for } k_0 > 0$$

$$f'_{\mathrm{Rst}}(-k_0, \boldsymbol{k}^2) = f'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)^*$$

we get from (45) the complete connection between the retarded functions f_{Ret} and f'_{Ret}:

(46) $$f_{\mathrm{Ret}}(p, k) = \frac{1}{(k+p)^2 + M^2 - i\varepsilon k_0}\,\frac{1}{(k-p)^2 + M^2 - i\varepsilon k_0}\,f'_{\mathrm{Ret}}(p, k)\,,$$

$$= \frac{1}{(k+p)^2 + M^2 - i\varepsilon(k_0 + m)}\,\frac{1}{(k-p)^2 + M^2 - i\varepsilon(k_0 - m)}\,f'_{\mathrm{Ret}}(p, k) \qquad \varepsilon \to +0\,,$$

because

$$\frac{1}{(k \pm p)^2 + M^2 - i\varepsilon k_0} = \frac{1}{(k \pm p)^2 + M^2 - i\varepsilon(k_0 \pm m)} \qquad \text{for} \qquad m < M\,.$$

On the energy shell $k_0 = \omega$, $\boldsymbol{k}^2 = \omega^2 - M^2$ equ. (39) becomes

$$f_{\mathrm{Ret}}(\omega, \omega^2 - M^2) = -\frac{1}{4m^2}\,\frac{1}{(\omega + i\varepsilon)^2 - (m^2/4)}\,f'_{\mathrm{Ret}}(\omega, \omega^2 - M^2)\,.$$

Since both factors on the right hand side can be continued regularly into the upper half plane of ω also $f_{\mathrm{Ret}}(\omega, \omega^2 - M^2)$ has an analytical continuation,

which is regular in the upper half plane of ω. This leads to the dispersion relation [19]

$$\text{(47)}\qquad \operatorname{Re} f_{\mathrm{Ret}}(\omega, \omega^2 - M^2) = a + \frac{2}{\pi}\int_0^\infty \mathrm{d}\omega' \, \frac{\omega' \operatorname{Im} f_{\mathrm{Ret}}(\omega', \omega'^2 M^{-2})}{\omega'^2 - \omega^2} =$$

$$= a + \frac{c}{\omega^2 - (m^2/4)} + \frac{2}{\pi}\int_M^\infty \mathrm{d}\omega' \, \frac{\omega' \operatorname{Im} f_{\mathrm{Ret}}(\omega'^2, \omega'^2 - M^2)}{\omega'^2 - \omega^2},$$

with

$$\text{(47')}\qquad c = -\frac{1}{4m^2} f'_{\mathrm{Ret}}\left(\frac{m}{2}, \frac{m^2}{4} - M^2\right).$$

The constant c can easily be expressed by the retarded function in the physical domain. Using for instance Oehme's method [14] for the continuation of f'_{Ret} we obtain

$$\text{(47'')}\qquad c = \frac{1}{4m^2}\int \mathrm{d}_3\boldsymbol{\xi} \exp\left[-i\boldsymbol{k}\boldsymbol{\xi}\right]\int \mathrm{d}\xi_0 \exp\left[i\frac{m}{2}\xi_0\right] h(\xi_0, \boldsymbol{\xi}'^2),$$

with

$$\boldsymbol{k}'^2 = \frac{m^2}{4} - M^2,$$

and

$$h(\xi_0, \boldsymbol{\xi}^2) = \left[\left(i\frac{\partial}{\partial\xi} + p\right)^2 + M^2\right]\left[\left(i\frac{\partial}{\partial\xi} - p\right)^2 + M^2\right]\cdot$$

$$\cdot\left\{\Phi_p, \left[R\psi\left(\frac{\xi}{2}\right)\overline{\psi}\left(-\frac{\xi}{2}\right) - \left\langle R\psi\left(\frac{\xi}{2}\right)\overline{\psi}\left(-\frac{\xi}{2}\right)\right\rangle_0\right]\Phi_{,}\right\}.$$

2) Another formula which expresses c in terms of the vertex function was given by SYMANZIK [17]. Using equ. (28) of Sect. **4** we can obtain SYMANZIK's result by a slightly different derivation of (47). According to (28) we have

$$\text{(48)}\qquad \left((k+p)^2 + M^2\right)\tilde{\eta}(p, k; p', k') =$$

$$= 2\pi i\left((k+p)^2 + M^2\right)\tilde{\eta}(p, k, -p-k)\left((k+p)^2 + M^2\right)\tilde{\eta}(p', k', -p-k),$$

if

$$(k+p)^2 + M^2 = 0$$

and

$$\text{(48')}\qquad \left((k+p')^2 + M^2\right)\tilde{\eta}(p, k; p', k') =$$

$$= 2\pi i\left((k+p')^2 + M^2\right)\tilde{\eta}(p', k; -k-p')\left((k+p')^2 + M^2\right)\tilde{\eta}(p, k'; -k-p),$$

---

[19] In this form the dispersion relation holds only if $f_{\mathrm{Ret}}(\omega, \omega^2 - M^2)$ is sufficiently bounded for $|\omega| \to \infty$. Compare references [13] until [18].

if

$$(k + p')^2 + M^2 = 0 .$$

The η-function $\tilde{\eta}(p, k; q)$ is defined by

$$\delta(p + k + q)\,\tilde{\eta}(p, k; q) = \tau(p, k, q) = \tag{49}$$

$$= \frac{1}{(2\pi)^{15/2}} \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \exp\left[-i(kx + qy + pz)\right] (\Omega, T\psi(x)\bar{\psi}(y)\,A(z)\Omega) .$$

For forward scattering (48) and (48′) become

$$f'(p, k) = i(2\pi)^4 \left((k - p)^2 + M^2\right) \Gamma(p^2, k^2, (k + p)^2)^2 \tag{50}$$

for

$$(k + p)^2 + M^2 = 0$$

and

$$f'(p, k) = i(2\pi)^4 \left((k + p)^2 + M^2\right) \Gamma(p^2, k^2, (k - p)^2)^2 \tag{50′}$$

for

$$(k - p)^2 + M^2 = 0$$

where the vertex function Γ is defined by

$$\Gamma(p^2, k^2, (p + k)^2) = -\frac{i}{(2\pi)^2} \left((k + p)^2 + M^2\right)\left((k - p)^2 + M^2\right) \cdot \tag{51}$$

$$\cdot \int \mathrm{d}\xi \exp\left[-i(k + (p/2))\xi\right] \left(\Phi_p, T\psi\left(\frac{\xi}{2}\right)\bar{\psi}\left(-\frac{\xi}{2}\right)\Omega\right),$$

or

$$\delta(p + k + q)\,\Gamma(p^2, k^2, q^2) = (p^2 + m^2)(k^2 + M^2)(q^2 + M^2)\,\tau(p, k, q) .$$

The corresponding retarded vertex function is

$$\Gamma_{\mathrm{Ret}}(p, k) = -\left((p + k)^2 + M^2\right)\left((p - k)^2 + M^2\right) \cdot \tag{52}$$

$$\cdot \int \mathrm{d}\xi \exp\left[-i\left(k + \frac{p}{2}\right)\xi\right] \left(\Phi_p, R\psi\left(\frac{\xi}{2}\right)\bar{\psi}\left(-\frac{\xi}{2}\right)\Omega\right).$$

On the energy shell $k_0 = \omega$, $\boldsymbol{k}^2 = \omega^2 - M^2$ the vertex function $\Gamma_{\mathrm{Ret}}(\omega, \omega^2 - M^2)$ has an analytical continuation [14-18] into the upper half plane because

$$\mathrm{Im}\,\Gamma_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0 \qquad \text{for } |k_0| < M.$$

We write now the forward scattering amplitude in the form

$$f_{\mathrm{Ret}}(p, k) = F_{\mathrm{Ret}}(p, k) + \Lambda(p, k)\,, \tag{53}$$

where

$$\Lambda(p, k) = \frac{\Gamma_{\mathrm{Ret}}(p, k)^2}{(k+p)^2 + M^2 - i\varepsilon(k_0 + m)} + \frac{\Gamma^*_{\mathrm{Ret}}(p, -k)^2}{(k-p)^2 + M^2 - i\varepsilon(k_0 - m)}\,, \tag{54}$$

$$F_{\mathrm{Ret}}(p, k) = \frac{1}{(k+p)^2 + M^2 - i\varepsilon(k_0 + m)} \cdot \frac{1}{(k-p)^2 + M^2 - i\varepsilon(k_0 - m)} F'_{\mathrm{Ret}}(p, k)\,, \tag{55}$$

$$F'_{\mathrm{Ret}}(p, k) = f'_{\mathrm{Ret}}(p, k) - ((k-p)^2 + M^2)\Gamma_{\mathrm{Ret}}(p, k)^2 - ((k+p)^2 + M^2)\Gamma^*_{\mathrm{Ret}}(p, -k)^2\,. \tag{55'}$$

We investigate the analytical properties of F_{Ret} and Λ separately and will show that the relations

$$\mathrm{Re}\, F_{\mathrm{Ret}}(\omega, \omega^2 - M^2) = a + \frac{2}{\pi}\int_M^\infty \mathrm{d}\omega'\, \frac{\omega'\, \mathrm{Im}\, F_{\mathrm{Ret}}(\omega', \omega'^2 - M^2)}{\omega'^2 - \omega^2}\,. \tag{56}$$

$$\mathrm{Re}\, \Lambda(\omega, \omega^2 - M^2) = \frac{c}{\omega^2 - (m^2/4)} + \frac{2}{\pi}\int_M^\infty \frac{\omega'\, \mathrm{Im}\, \Lambda(\omega', \omega'^2 - M^2)}{\omega'^2 - \omega^2}\,, \tag{57}$$

hold which lead immediately to a dispersion relation for $f'_{\mathrm{Ret}}(\omega, \omega^2 - M^2)$.

a) Proof of relation (56). – The function $F'_{\mathrm{Ret}}(p, k)$ has, for $|k_0| < M$, the properties

$$\left.\begin{array}{ll} \text{(i)} & \mathrm{Im}\, F'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0 \\ \text{(ii)} & F'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0 \quad \text{for } (k-p)^2 + M^2 = 0 \\ \text{(iii)} & F'_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0 \quad \text{for } (k+p)^2 + M^2 = 0 \end{array}\right\} \; |k_0| < M\,,$$

(ii) and (iii) follow for $|k_0| < M$ immediately from (50), (50') and (55'). $F_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)$ satisfies therefore

$$\mathrm{Im}\, F_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2) = 0 \qquad \text{for } |k_0| < M.$$

Since furthermore the Fourier transform of $F_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)$ is a retarded function and $F_{\mathrm{Ret}}(-k_0, \boldsymbol{k}^2) = F^*_{\mathrm{Ret}}(k_0, \boldsymbol{k}^2)$ we obtain relation (56).

b) *Proof of relation* (57). – On the energy shell $\Lambda(k_0, \boldsymbol{k}^2)$ becomes

$$\Lambda(\omega, \omega^2 - M^2) = \frac{1}{2m} \frac{\Gamma^*_{\mathrm{Ret}}(-\omega, \omega^2 - M^2)^2}{\omega - (m/2) + i\varepsilon} - \frac{1}{2m} \frac{\Gamma_{\mathrm{Ret}}(\omega, \omega^2 - M^2)^2}{\omega + (m/2) + i\varepsilon}, \tag{58}$$

$\Lambda(\omega, \omega^2 - M^2)$ has therefore an analytical continuation which is regular in the upper half plane and it follows that

$$\mathrm{Re}\, \Lambda(\omega, \omega^2 - M^2) = \frac{2}{\pi} \int_0^{M} \mathrm{d}\omega' \frac{\omega' \,\mathrm{Im}\, \Lambda(\omega', \omega'^2 - M^2)}{\omega'^2 - \omega^2} =$$

$$= \frac{c}{\omega^2 - (m^2/4)} + \frac{2}{\pi} \int_M^{\infty} \mathrm{d}\omega' \frac{\omega' \,\mathrm{Im}\, \Lambda(\omega', \omega'^2 - M^2)}{\omega'^2 - \omega^2}.$$

with

$$c = \frac{1}{2} \Gamma_{\mathrm{Ret}} \left(-\frac{m}{2}, \frac{m^2}{4} - M^2 \right)^2. \tag{59}$$

The sum of (56) and (57) gives the final dispersion relation [19]

$$\mathrm{Re}\, f(\omega) = a + \frac{c}{\omega^2 - (m^2/4)} + \frac{2}{\pi} \int_M^{\infty} \mathrm{d}\omega' \frac{\omega' f(\omega')}{\omega'^2 - \omega^2}, \tag{60}$$

for the forward scattering amplitude

$$f(\omega) = f_{\mathrm{Ret}}(\omega, \omega^2 - M^2).$$

The constant c is given by (59) in agreement with Symanzik's result. $\Gamma_{\mathrm{Ret}}(-m/2, (m^2/4) - M^2)$ may be written in the form [17]

$$\frac{1}{(2\pi)^2} \Gamma_{\mathrm{Ret}} \left(-\frac{m}{2}, \frac{m^2}{4} - M^2 \right) = f + \frac{1}{\pi} \int_{-\infty}^{-(m+M)^2} \mathrm{d}q^2 \frac{\mathrm{Im}\, \Gamma(-M^2, -m^2, q^2)}{-q^2 - M^2}.$$

where the vertex function Γ is defined by (51).

* * *

I should like to thank Dr. R. KARPLUS for the kind hospitality of the Physics Department, University of California.

RIASSUNTO (*)

Si studiano le singolarità delle funzioni di Green derivanti dal vuoto intermedio e dagli stati ad una particella

(*) *Traduzione a cura della Redazione.*

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Some Results on the Photodisintegration of Samarium.

G. Di Caporiacco and M. Mandò

Istituto di Fisica dell'Università - Firenze
Istituto Nazionale di Fisica Nucleare - Sottosezione di Firenze

F. Ferrero

Istituto di Fisica dell'Università - Torino
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

(ricevuto il 12 Marzo 1959)

Summary. — The total electron capture to β^+ emission of ^{143}Sm has been determined, with respect to that of ^{140}Pr; the experimental result is $\varepsilon_{Sm} = \varepsilon_{Pr} \cdot 1.289$. Taking this branching ratio into account the yield of the $^{144}Sm(\gamma, n)$ reaction is shown to be in agreement with the theoretically expected value. Evidence is given for a 74^m activity, attributable to ^{142}Sm from the $^{144}Sm(\gamma, 2n)$ reaction; an indication for its $K : \beta^+$ ratio is also given.

1. – Introduction.

The photodisintegration of Samarium was first observed by Butement [1], who irradiated a natural Samarium sample (as Sm_2O_3) with 22 MeV bremsstrahlung γ-rays; he was able to detect in the irradiated sample a β activity of about 8 minutes, which was reasonably ascribed to ^{143}Sm. The same activity was also produced by fast neutron bombardment of natural Sm_2O_3 by Mirnik and Aten [2], wo ascribed it to (n, 2n) reaction, thus comfirming the previous assignment of the activity to ^{143}Sm. They found a half life of 8.3 min, a maximum β energy of (2.3 ± 0.3) MeV (by an absorption method) and did not detect any γ-rays with energy > 150 keV.

(2) F. D. Butement: *Proc. Phys. Soc.*, A **64**, 395 (1951); *Nature*, **165**, 149 (1950).
(1) M. Mirnik and A. H. W. Aten Jr.: *Physica*, **22**, 14 (1956).

The most exhaustive study, however, was carried out by SILVA and GOLDEMBERG ([3]), who were able to confirm definitely the β^+ character of the activity, to measure accurately the half-life (for which they find a value of 9.03 min by running their betatron below the oxigen threshold) and to give the maximum β^+ energy as 2.6 MeV, by an absorption method.

Finally, these authors measured the yield of the (γ, n) reaction up to 22 MeV and deduced the cross-section by the photon difference method. The yield was surprisingly low, by about a factor four, as compared to the theoretical estimates and a reasonable interpolation between other known data.

This discrepancy, together with the fact that photodisintegration of ^{144}Sm could lead in « a island of isomerism », as pointed out by SILVA and GOLDEMBERG ([3]), lead us to take up again the question with the 30 MeV betatron of the University of Turin. Although, at the present time, the question cannot be considered as completely settled, we believe that some of the results we have so far obtained are of sufficient interest to warrant publication.

They include: *a*) experimental detection (in Samarium samples irradiated with 22 MeV and 30.5 MeV maximum bremsstrahlung energy) of X-rays of the same energy expected for K capture in Samarium, with a 9 min half-life; *b*) a measurement of the $K{:}\beta^+$ ratio for the 9 min activity; *c*) some values of the yield ratio $^{143}Sm/^{140}Pr$ at 22 MeV and at 30 MeV, $^{140}Pr/^{63}Cu$ at 30 MeV; *d*) the detection of a 74 min half-period activity, which can be reasonably ascribed to ^{142}Sm (probably K and β^+) ([4]).

2. – Experimental set-up.

Samples of Pr_6O_{11} and Sm_2O_3 (99.9 % purity) were irradiated at betatron energies of 22 MeV and 30.5 MeV and then counted with a NaI scintillator as specified below; several runs were made, with essentially two different irradiation geometries (a distant exposure with a well collimated beam, and a near exposure without collimation), but in each case parallel runs for Pr_6O_{11} and Sm_2O_3 were obtained with the same geometry.

The irradiated samples, in the form of powders, were spread and pressed as uniformly as feasible in polistyrene dishes of 38 mm diameter; the main

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([3]) E. SILVA and J. GOLDEMBERG: *Nuovo Cimento*, **3**, 12 (1956).

([4]) While the present results were being elaborated the Table of Isotopes by D. STROMINGER, J. M. HOLLANDER, and G. T. SEABORG: *Rev. Mod. Phys.*, **30**, 585 (1958), came to our kowledge; here an activity of 72^m is reported as a private communication from T. V. MARSHALL and J. O. RASMUSSEN and ascribed to ^{142}Sm. Our results give independent confirmation of this activity; a comparative discussion of the two results is contained in the last paragraph.

group of measurements were carried out with a mass of about 1.2 g, both for Pr_6O_{11} and Sm_2O_3, but auxiliary measurements were also made with masses as low as 0.6 g and as high as 2.2 g in order to estimate the correction for self absorption.

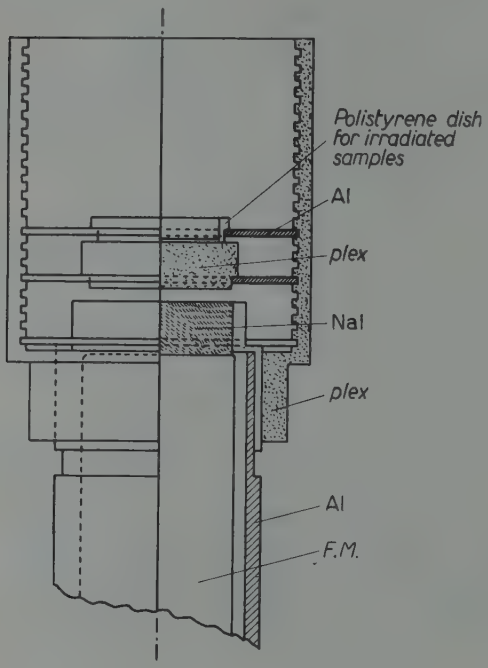

Fig. 1. – Counting geometry.

The counter was essentially a NaI crystal $\frac{1}{2}$ in. thick and $1\frac{1}{2}$ in. in diameter (with a Du Mont 6292 photomultiplier); a plexiglass absorber prevented β-rays from reaching the detector and at the same time represented the main and reasonably well localized source of annihilation γ-rays; the counting geometry with plexiglass is shown in Fig. 1.

The pulses from the photomultiplier were fed to two conventional amplifiers, the first one driving a five channel amplitude analyzer and the second one a simple discriminator.

The threshold voltage of the five channel analyzer was adjusted so as to have the measured X line centered at the middle channel; for calibration the well known 46.5 keV γ line of RaD was used ([5]); the channel amplitude was selected so as to include a large fraction of the counts of the line in the five channels (see example in Fig. 2).

The discriminator threshold was adjusted indirectly, by suitably stepping-down the amplification, to values which could possibly be different for different runs (though the same for Pr_6O_{11} and Sm_2O_3 of course) but in any case excluded low energy γ-rays, as the main γ's of the simultaneously occurring 47 h half life ^{153}Sm.

In this way it was assumed that counts in the discriminator (which we will call D channel) could arise only from annihilation γ of the β^+, mainly in the plexiglass absorber. Essentially two values for the threshold were used,

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([5]) See *e.g.* R. W. FINK: *Phys. Rev.*, **106**, 266 (1957); M. FRILLEY and M. VALDARES: *Journ. Phys. Rad.*, **18**, 468 (1957); G. T. EWAN and M. A. S. ROSS: *Nature*, **170**, 760 (1952).

corresponding approximately to 125 keV and to 400 keV and no difference was observed in the ratio Sm/Pr.

A ionization chamber, of the type described by FERRERO *et al.* (6), served as a monitor for the beam intensity in every single run.

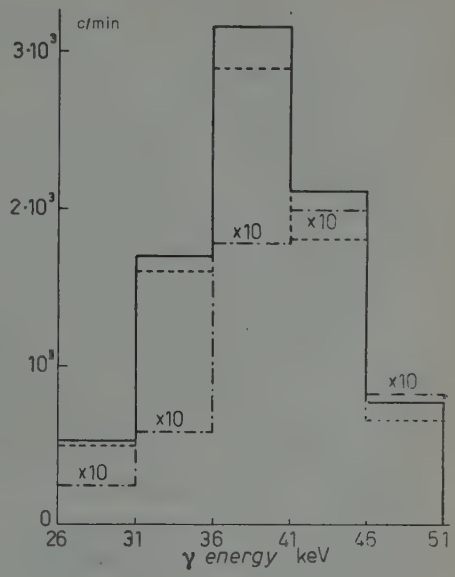

Fig. 2. – Five channel spectrum of the K-X-line ——— uncorrected ^{143}Sm plus ^{153}Sm; ---- ^{143}Sm corrected for ^{153}Sm; – · – · – ^{153}Sm only. The shift between the Pm-K-line following capture in ^{143}Sm and the Eu-K-line due to conversion of gammas following β^--decay of ^{153}Sm is clearly apparent.

3. – K/β^+ branching of ^{143}Sm.

Let us call ε the total capture to β^+ emission ratio and k the K shell capture to β^+ emission ratio; according to BRYSK and ROSE's calculations, as quoted in the Tables 5.41 of NIJGH, WAPSTRA and VAN LIESHOUT (7), ε/k is a slowly varying function of Z; its value for Pr ($Z = 59$) is 1.156 and for Sm ($Z = 62$) is 1.162. The fluorescent yield in the K shell (Table 7.2-3 of l. c. (7)) is for Ce (daughter of ^{140}Pr) 0.895 and for Pm (daughter of ^{143}Sm) 0.908. These values will be used whenever necessary in what follows without further reference.

Although the assignment of the 9.03 min activity to ^{143}Sm cannot be considered as definitely established, it seems that at present it cannot be reasonably questioned; any other assignment would render the interpretation of existing evidence more difficult. On the other hand none of the detectors used by previous authors was sensitive to X-rays following electron-capture. With a 2.6 MeV β^+ end-point energy the expected K/β^+ ratio for Samarium ($Z = 62$) for allowed transition is 0.802 (8). This reduces the anomaly observed in the yield

(6) F. FERRERO, R. MALVANO and C. TRIBUNO: *Nuovo Cimento*, **5**, 510 (1957).

(7) G. J. NIJGH, A. H. WAPSTRA and R. VAN LIESHOUT: *Nuclear Spectroscopy Tables* (Amsterdam, 1959).

(8) P. F. ZWEIFEL: *Phys. Rev.*, **96**, 1572 (1954), later corrected in *Phys. Rev.*, **107**, 329 (1957). Cfr. also (7), p. 64.

in [3] although it cannot account for the whole of it; in view of the sensitiveness of this ratio to the exact value of the energy and our ignorance of the possible degree of forbidness of the transition, it was decided to measure directly this ratio, by comparing it to the corresponding ratio of ^{140}Pr which is experimentally well known [9].

Eight runs (4 with Pr and 4 with Sm) were made at 22 MeV for this purpose; the counts were corrected for background and reduced to unit time; Praseodymium was then corrected for residual activity and reduced to zero time (end of exposure) with 3.4 min half-life, and to saturated activity and unit beam intensity and unit mass of the sample. Measurements, each of two minutes duration, were taken, at different times from 8 to 20 minutes after exposure; within this range the statistical error of a single measurement for Praseodymium was less than 1% and no need for oxigen correction was apparent; for Samarium the same corrections were applied (residual activity being considerable, coming essentially from 47 h ^{153}Sm); statistical errors were here up to 1.5%. Internal consistency within each run was of the same order; as an index I_K for the intensity of the K-X line the sum of the counts in the five channels, corrected as above, was taken; as an index I_β for the β^+ intensity the counts above the discriminator threshold were taken.

The following values were obtained in the first six runs for the ratio $J = I_k : I_\beta$: Praseodymium $J = 1.998$; 1.954; 1.945 (average 1.969); Samarium $J = 2.672$; 2.534; 2.549 (average 2.585).

Since geometrical irradiation and detection conditions were the same and assuming that detection efficiency is also the same, we get, taking into account the fluorescence yield,

$$k_{\mathrm{Sm}} : k_{\mathrm{Pr}} = \frac{2.585}{0.908} : \frac{1.969}{0.895} = 1.294$$

and introducing the total to K capture ratios:

$$\varepsilon_{\mathrm{Sm}} : \varepsilon_{\mathrm{Pr}} = 1.163\, k_{\mathrm{Sm}} : 1.156\, k_{\mathrm{Pr}} = 1.301$$

The seventh and eighth run were carried out with a different threshold of the discriminator (125 keV instead of 400 keV for the previous runs) and gave, in a similar way, $k_{\mathrm{Sm}} : k_{\mathrm{Pr}} = 1.248$ and $\varepsilon_{\mathrm{Sm}} : \varepsilon_{\mathrm{Fr}} = 1.254$. As a grand average we therefore assume

(1) $$k_{\mathrm{Sm}} : k_{\mathrm{Pr}} = 1{,}283$$

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[9] C. I. Browne, J. O. Rasmussen, J. P. Surls and D. F. Martin: *Phys. Rev.*, **85**, 146 (1952). The ε value given by these authors is 0.724 but we will adopt insteated the value 0.852, as revised by J. O. Rasmussen, quoted as private communication in l.c. [4].

and

$$\varepsilon_{Sm} : \varepsilon_{Pr} = 1.289 . \tag{2}$$

If, now, we assume for ε_{Pr} the experimental value 0.852 (l. c. [9]) (and therefore $k_{Pr} = 0.737$) we get $\varepsilon_{Sm} = 1.098$ (and therefore $k_{Sm} = 0.945$, which corresponds, according to [8], to a maximum β energy of 2.46 MeV), if, instead, we assume the theoretical value [8] for ^{140}Pr, namely $k_{Pr} = 0.867$, we get $k_{Sm} = 1.112$ which would correspond to a maximum β^+ energy of 2.33 MeV.

As above said the statistical error should be less than 1%, but internal consistency suggests that the value (1) might be affected by a probable error of about 2%. Numerous sources of systematic error may be present, but a careful and conservative estimate of possible causes of systematic error lead us to believe that their amount can hardly exceed 5%. Among these the possible effect of the different detection efficiency for annihilation γ-rays, due to the slightly different β^+ energies of ^{143}Sm and ^{140}Pr and the consequentially different geometrical distribution of annihilation radiation, has also been considered; its contribution should not exceed $2 \div 3$%, as can be gathered from the fact that the yield ratio Sm/Pr comes out the same with a substantially different geometry (see Section **4**).

Another source of possible systematic error is self-absorption of K-X rays; an experimental estimate of the correction needed was made with Praseodymium; it came out that the reduction to zero thickness would imply a correction as high as 40%, but, since the conditions are quite similar for Pr and Sm and care was taken to use the same mass for Pr and Sm, no correction was applied, assuming that the two corrections would cancel out; the direct determination of the self absorption in Samarium, would be more uncertain, owing to the numerous corrections required.

We believe that the present method of measuring the K/β^+ ratio, with minor modifications, may prove useful in many other cases of low activity, whenever a neighbouring substance is available for calibration.

4. – Relative yields of ^{144}Sm (γ, n), ^{141}Pr(γ, n) and ^{63}Cu(γ, n) reactions.

On the basis of the same runs already used for the K/β^+ determination, the yield [10] ratio at 22 MeV of ^{144}Sm to ^{141}Pr comes out $Y(\mathrm{Sm})/Y(\mathrm{Pr}) = 1.088$. The measurements at 30 MeV could not be used for K/β^+ determination be-

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([10]) Defined as the ratio of saturated activity (disintegration/min) in the product nuclide (say ^{143}Sm) per mole of bombarded nuclide (say ^{144}Sm) and per unit intensity (röntgen/min); in comparing results at different energies we reduced our standard monitor, l.c. ([6]), with Al, to the Plexiglass standard ([11]) unless otherwise stated.

([11]) L. Katz and A. C. Cameron: *Can. Journ. Phys.*, **29**, 518 (1951).

cause, after applying the various corrections for ^{15}O, ^{153}Sm, and 74 min activities (see Section 5), apparently some low activity, other than 9 min, still remained in the D channel. The measurements at 30 MeV can however be used for the yield ratio, provided the previously determined ε is considered correct and the index I_K is taken as the basis for the comparison; within the 1% estimated statistical error the ratio Sm/Pr comes out the same at 30.5 MeV as at 22 MeV; the yield of both Pr and Sm at 30.5 MeV, taking into account the response of the monitor, is about 7% higher than at 22 MeV; this latter figure is, however, more uncertain.

It is worth nothing that, although Samarium irradiated at 30.5 MeV shows apparently an index I_β which grows with the time of measurement after exposure, the various runs are, at a given time, consistent with each other. Even using a different geometrical arrangement by enclosing the irradiated sample within a 2 mm brass absorber, instead of the Plexiglass absorber, thus suppressing the K line and getting a more concentrated source of annihilation γ's, the ratio Sm/Pr at a given time remains the same as with the Plexiglass absorber within 2%. This gives some confidence in the method.

Many irradiations were also made at 30.5 MeV with Cu, and counts taken both with Plexiglass and with brass absorbers. As an average we get for the yield ratio $Y_{Pr}/Y_{^{63}Cu}$ a value of 3.79 (evaluated with $\varepsilon_{Pr} = 0.852$) with an internal consistency of about 5%; besides various sources of systematic error may add perhaps another 5%. Within these limits, the above value (3.79) fits well within the general trend of the yield as a function of atomic number Z, as given for instance by PRICE and KERST [12]; besides it is only 6% lower than a more accurate value by FERRERO and SILVA (private communication) obtained with a substantially different procedure.

As for ^{144}Sm, on the basis of the above ratio Sm/Pr = 1.088, our value for the absolute yield, though still somewhat low, as compared to the general trend of PRICE and KERST, does not show however any sharp disagreement with the theory, being within the combined experimental error and uncertainty of the theoretical estimate.

5. – The 74^m activity.

When irradiating at 30.5 MeV, the corrections were considerably more important than at 22 MeV. For Praseodymium, however, only two minor corrections (for ^{15}O and ^{139}Pr) were sufficient to make the data from $t = 8$ min to $t = 35$ min consistent with the single expected half-life of 3.4 min ^{140}Pr.

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[12] G. PRICE and D. W. KERST: *Phys. Rev.*, **77**, 806 (1950).

They were evaluated with sufficient approximation as follows: *a*) for ^{15}O on the basis of dosis and the result of a separate irradiation of water and *b*) for ^{139}Pr from (γ, 2n) reaction, by measuring the residual activity about 1 h or 2 h from the Pr_6O_{11} exposure.

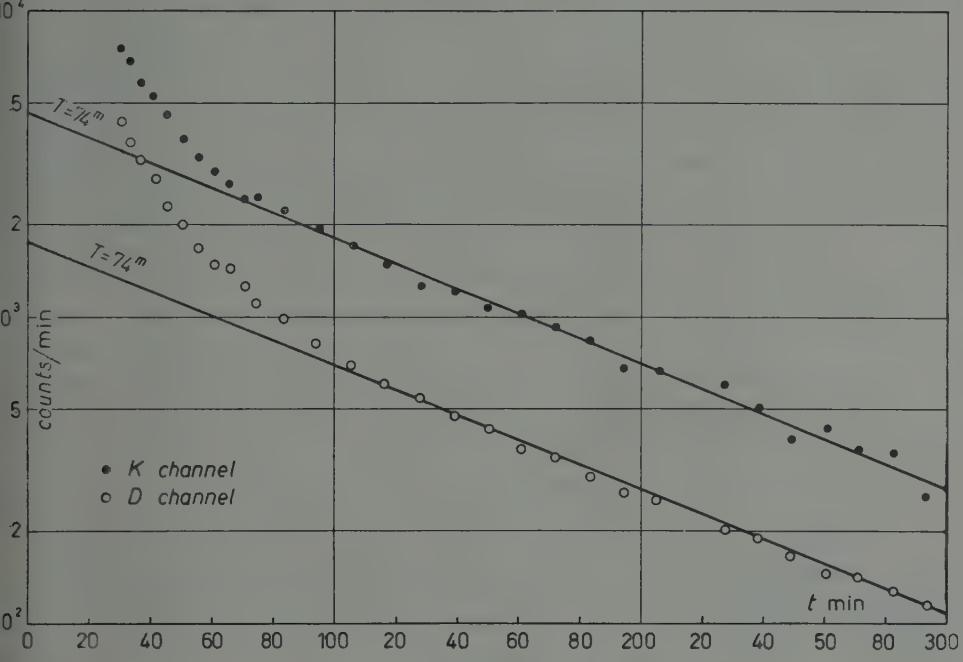

Fig. 3. – Decay curve of Sm irradiated at 30.5 MeV after correction for ^{153}Sm 47 h activity; fluctuation in K channel is bigger owing to the considerable amount of correction.

For Samarium it was soon apparent that after the ^{15}O and ^{153}Sm (47 h) corrections, the 9 min activity was not pure, but a longer half life was still present. Some special runs were then taken with 90 min or 120 min irradiation time (the usual irradiation time being instead 9 min or 18 min). Fig. 3 shows a typical decay curve observed; after correcting for the residual 47 h activity, the presence of a 74 min activity is clearly apparent, both in the *K* channel and in the *D* channel.

This activity can be reasonably ascribed to ^{142}Sm from ^{144}Sm (γ, 2n) reaction, no other activity of comparable half life being known among those to which the photodisintegration of Sm could lead; the assignment is consistent with its low yield at 30 MeV (the exact value cannot be stated without definite assumptions about its decay scheme) and with the fact that by re-

ducing the maximum bremsstrahlung energy to 22 MeV, the yield becomes almost negligible.

While these results were being elaborated we learned from the Table of Isotopes in the April issue of *Rev. Mod. Phys.* (1958) ([4]) that a 72 min activity has been ascribed to ^{142}Sm in a private communication by MARSHALL and RASMUSSEN to STROMINGER *et al.* ([4]); according to these authors ^{142}Sm would decay to a ~ 30 s half-life ^{142}Pm with β^+ of 3.78 MeV end point energy.

By analyzing our measurements of the 74 min activity as above done for ^{143}Sm (9 min activity) we get for $J = I_k : I_\beta$ about the same value as for ^{143}Sm; if, therefore, the counts were only due to a supposed K/β^+ branching in ^{142}Sm (thus neglecting ^{142}Pm altogether, as it would be the case for a very long half life) we would get an apparent ratio $\varepsilon_{app} \simeq 1.10$ or $\simeq 1.28$, according to whether $\varepsilon_{Pr} = 0.852$ or $k_{Pr} = 0.867$ is accepted for ^{140}Pr.

On the basis of the data of MARSHALL and RASMUSSEN, our result could tentatively be reinterpreted as follows. The 3.78 MeV β^+ end point energy of ^{142}Pm would correspond to $k_{1\,m} = K/\beta^+ = 0.24$ (from Zweifel's theoretical tables). Let us assume that no γ-rays, other than annihilation γ's, are present in the D channel (threshold 400 keV); let us also assume for D channel and K channel the same sensitivity as for Pr. By keeping in mind that ^{142}Sm and ^{142}Pm, being in equilibrium, must have the same disintegration rate, we get the *true* value k (^{142}Sm) $\simeq 3.5$ if $k = 0.737$ ($\varepsilon = 0.852$) is assumed correct for ^{140}Pr, or k (^{142}Sm) $\simeq 6.5$ if $k = 0.867$ is assumed for ^{140}Pr. The maximum β^+ energy would vary correspondingly from $\simeq 1.55$ MeV to $\simeq 1.25$ MeV.

Although the above deductions are perfectly consistent and the assumptions quite reasonable, their tentative character can hardly be overemphasized.

6. – Conclusions.

In conclusion our measurements give for ^{143}Sm a value of the total electron capture to β^+ emission branching ratio which is 1.29 times the corresponding value for ^{140}Pr; measurements of the relative yields of ^{143}Sm from ^{144}Sm (γ, n) and ^{140}Pr from ^{141}Pr (γ, n) with respect to ^{63}Cu (γ, n) fit reasonably well the general trend of the yields as a function of Z and the theoretical estimates; finally the 72 min activity reported by MARSHALL and RASMUSSEN ([4]) is confirmed and a raw indication is given for its K/β^+ ratio, from which a more reliable value could be obtained, when more were known about the decay scheme of both ^{142}Sm and its daughter ^{142}Pm.

* * *

Finally we would like to thank Prof. G. WATAGHIN and Prof. S. FRANCHETTI, whose encouraging help enabled the collaboration between the two laboratories. Thanks are also due to Prof. G. CANNERI and Prof. G. PICCARDI of the Chemistry Dept. of the University of Florence for supplying us with Pr and Sm samples of the highest purity. It is a great pleasure to acknowledge the friendly and useful discussions with Dr. E. SILVA and the friendly collaboration kindly given us by Dr. M. BOCCIOLINI in the collection and elaboration of the experimental data; our sincere thanks are also due to the whole staff of the Turin betatron and particularly to Dr. Ing. GONELLA and Mr. G. MICHELETTA.

RIASSUNTO

Il rapporto di diramazione tra cattura elettronica ed emissione β^+ per il ^{143}Sm viene misurato rispetto a quello del ^{140}Pr; risulta $\varepsilon_{Sm} = \varepsilon_{Pr} \cdot 1.289$ cioè, ammettendo $\varepsilon_{Pr} = 0.852$ (valore sperimentale di Browne *et al.*) $\varepsilon_{Sm} = 1.098$. Se, invece, si accetta per il ^{140}Pr il valore teorico $\varepsilon_{Pr} = 1.002$ (calcolato in base all'energia dalle tabelle dello Zweifel e al rapporto 1.162 della cattura totale alla cattura K da Brysk e Rose) risulta dalle nostre misure $\varepsilon_{Sm} = 1.292$. Viene messa in evidenza un'attività di periodo di dimezzamento 74^m, attribuibile al ^{142}Sm da $(\gamma, 2n)$ su ^{144}Sm e viene data un'indicazione sul valore del suo rapporto di diramazione K/β^+.

Relativistic Particle with Internal Rotational Structure.

T. TAKABAYASI (*)

Institut Henri Poincaré - Paris

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Summary. — General kinematical and dynamical aspects of any covariant theory of particles possessing internal angular velocity together with internal angular momentum are analysed. Theory is classified according to the form of subsidiary conditions required. A precise theory is constructed by the aid of a spinor ζ as the kinematical representation of such structure, postulating the equation of motion,

$$2iQ\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} - ip_\mu\gamma_\mu\zeta - (m'^2/m)\{(\bar{\zeta}\zeta)\zeta - (\bar{\zeta}\gamma_5\zeta)\gamma_5\zeta\} = 0\,,$$

where ζ is defined along a world line, Q and m represent the structure constants of the original particle, while m' means observable rest mass which can take a certain constant value between zero and m. The equation contains a self-interaction term of the third order, and thus the theory may be considered to represent the particle analogue to the non-linear field of Heisenberg. Still it can be solved exactly, the result showing that the particle performs, besides its mean rectilinear motion represented by the constant momentum vector p_μ, an orbital zitterbewegung around it, and the synchronized precession of the internal third axis around the constant helicity pseudovector. The theory of Weyssenhoff is included in the present theory as a degenerate case in which the internal angular velocity becomes physically arbitrary. It is expected that a unified model of elementary particles may be reached by quantization starting from this generalized classical model.

1. – Introduction and general considerations.

As is well-known, it is difficult to construct a relativistic theory of a body with internal rotational degrees of freedom in a simple and unambiguous manner, in so far as the body must have finite spatial extension. This is related

(*) Now at the Physics Institute, University of Nagoya.

to the fact that the concept of rigid body, which could limit the internal degrees of freedom of the body to those of pure rotation, becomes ambiguous in relativity. However, this difficulty disappears if we imagine a local body of infinitesimally small spatial dimensions, still preserving the internal rotational degrees of freedom, that is, a « *particle rotator* ».

A quite interesting theory of this kind was formulated by WEYSSENHOFF [1]. His theory, however, corresponds just to a particular case in which the degrees of internal rotation degenerate and only the internal angular momenta are retained as physical variables.

In this paper we shall first treat from a general point of view the theory of the particle rotator in the framework of special relativity. Especially we shall establish the kinematics of such an entity and clarify the physical meaning of the 4-dimensional internal angular velocity. We show that such a particle is described in a most concise way by a spinor ζ defined along a world line. On the other hand we shall make clear the general dynamical aspect of such a particle related to the conservation laws of momentum-energy and angular momentum, which lead to the differentiation between true rest mass and observable rest mass and the conservation of the « helicity vector ».

We next proceed to construct a precise formulation of a theory of the particle rotator by postulating a definite form for the equation of motion. This we determine by the aid of a certain analogy with the Dirac theory of the electron so that it would fit the aim of eventually supplying a new possibility of elementary particle model.

For simplicity we treat the theory for the case of free particle. Also the theory is developed in this paper in its classical level (although we employ a spinor for the kinematical description of the particle), while the quantization will be investigated in subsequent works,

1. 1. *Conservation laws, disparity between true rest mass and observable rest mass.* – The particle motion has to satisfy the conservation laws of momentum-energy and angular momentum. First, momentum-energy 4-vector p_μ must be assigned to the particle, satisfying.

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} \equiv \dot{p}_\mu = 0 \,, \quad i.e., \; p_\mu = \text{constant of motion}, \tag{1}$$

where τ denotes the proper time. On the other hand the 4-velocity of the particle [2] $v_\mu = \dot{x}_\mu$ is restricted by

$$v_\mu^2 = -1 \,. \tag{2}$$

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[1] J. WEYSSENHOFF and A. RAABE: *Acta Phys. Pol.*, **9**, 7 (1947).

[2] In this paper we employ the unit system in which $c=1$. We use an imaginary time component, thus $x_4 = it$, and the energy of the particle is p_4/i.

The particle has internal angular momentum S_{ij} ([3]) which constitutes the space part of a six vector $S_{\mu\nu} = -S_{\nu\mu}$. The existence of the internal angular momentum is intimately related to the possible disparity between the velocity v_μ and momentum p_μ of the particle. The conservation law of angular momentum is represented by

$$\dot{S}_{\mu\nu} = p_\mu v_\nu - p_\nu v_\mu \,. \tag{3}$$

If in particular p_μ is colinear with v_μ, (3) reduces to $\dot{S}_{\mu\nu} = 0$; namely the internal and orbital parts of the angular momentum are separately conserved. However, this need not be the case in general.

The direct consequence of the disparity between p_μ and v_μ for a free particle is the existence of two different rest masses to be assigned to the particle. In the first place we take the rest frame of the particle, say Σ, namely the inertial frame in which the particle is momentarily at rest. In Σ we have

$$v_\mu^{(\Sigma)} = i\delta_{\mu 4} \,, \tag{4}$$

and hence the scalar invariant $-p_\mu v_\mu$ equals $p_4^{(\Sigma)}/i$, which signifies exactly the rest energy of the particle m, thus

$$-p_\mu v_\mu = m \,. \tag{5}$$

We assume this to be the particle's structure constant ([4]).

On the other hand one can write

$$p_\mu^2 = \text{const} = -m'^2 \tag{6}$$

on account of (1) and of the time-like (or null) character of p_μ which is physically required. If $m' \neq 0$ we can define a constant unitary vector,

$$\hat{p}_\mu = p_\mu / m' \,, \qquad (\hat{p}_\mu^2 = -1) \tag{7}$$

which differs from v_μ, representing the 4-velocity of the *mean* motion of the particle as we shall see later. We denote the rest frame of this mean motion as Π, in which

$$\hat{p}_\mu^{(\Pi)} = i\delta_{\mu 4} \,. \tag{8}$$

Then m′ signifies the energy of the particle in Π.

([3]) Greek indices run over 1 to 4 while latin indices over 1 to 3.

([4]) In the following we adopt positive m for simplicity, but we might equally assume m as a negative constant.

Writing (5) in the Π frame we get

$$|\boldsymbol{V}^{(\Pi)}| = (1 - m'^2/m^2)^{\frac{1}{2}} \equiv \beta \,, \tag{9}$$

where $\boldsymbol{V}_k^{(\Pi)} = iv_k^{(\Pi)}/v_4^{(\Pi)}$ represents the 3-dimensional velocity of the particle in the Π frame. The particle thus performs a precessional motion (« zitterbewegung ») around its constant mean rectilin ear motion, with the precessional of velocity a constant magnitude (9). (9) also indicates [5]

$$0 \leqslant m' = m(1 - \boldsymbol{V}^{(\Pi)2})^{\frac{1}{2}} \leqslant m \,. \tag{10}$$

Now the rest mass which is actually observed for an elementary particle corresponds not to m but to m', since empirically we obtain the rest mass through the relation (6) or by the inertial effect under an external force. We shall therefore call m as true rest mass and m' as observable (or apparent) rest mass of the particle. In contrast to m, we do not assume m' to be a particle constant. Namely, m' is merely a constant of motion, which can change from one possible motion to another, between two extreme values:

$$m' = m \,, \qquad \text{when} \quad |\boldsymbol{V}^{(\Pi)}| = 0 \,, \tag{11a}$$

$$m' \to 0 \,, \qquad \text{when} \quad |\boldsymbol{V}^{(\Pi)}| \to 1 \text{ (light velocity).} \tag{11b}$$

This suggests the possibility that the theory may lead to a mass spectrum distributing between 0 and m, corresponding to various quantum states of the same original particle with single true rest mass m, if the quantization of motion restricts the precessional velocity to certain discrete eigenvalues. Especially, the case (11a) corresponds to colinearity between v_μ and p_μ, and we then have

$$p_\mu = mv_\mu \,, \qquad \dot{v}_\mu = 0 \,, \qquad \dot{S}_{\mu\nu} = 0 \,. \tag{12}$$

The internal and external motions are there perfectly decoupled and the zitterbewegung disappears.

Here we remark the following: As is well-known, p_μ and v_μ also differentiate for a usual particle if an external electromagnetic field is present, such that

$$p_\mu = mv_\mu + eA_\mu \,. \tag{13}$$

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[5] At first sight (10) looks contradictory with the general relation, $m = m_0/(1 - \boldsymbol{V}^2)^{\frac{1}{2}}$ meaning the increasing of inertia with the particle velocity; however, this only implies that the latter relation does not apply to our present situation in which p_μ is not colinear to v_μ.

Furthermore it can be proved (6) that the motion of a free Weyssenhoff particle is identical with that of a spinless particle with unit charge under an external electromagnetic field of the constant strength:

$$F_{\mu\nu} = -(i/Q^2)\varepsilon_{\mu\nu\varkappa\lambda}s_\varkappa p_\lambda = Q^{-2}(mS_{\mu\nu} - v_{[\mu}N_{\nu]}) , \tag{14}$$

where s_μ and N_μ are the quantities defined by (15) and (21). However, such equivalence does not hold for the general case of particle rotator.

1·2. *Spin pseudovectors. Conservation of helicity.* – The six-vector $S_{\mu\nu}$ is mathematically equivalent to the pair of vectors

$$s_\mu = \tilde{S}_{\mu\nu}v_\nu , \tag{15}$$

$$t_\mu = S_{\mu\nu}v_\nu , \tag{16}$$

where $\tilde{S}_{\mu\nu} \equiv -(i/2)\varepsilon_{\mu\nu\varkappa\lambda}S_{\varkappa\lambda}$ denotes the dual of $S_{\mu\nu}$. The s_μ represents the pseudo-vector that coincides with the internal angular momentum in the rest frame Σ, such that

$$s_k^{(\Sigma)} = S_{ij}^{(\Sigma)} , \qquad s_4^{(\Sigma)} = 0 , \qquad (ijk) \sim (123) \quad (7) \tag{17}$$

and therefore it is suitably called the *spin pseudovector*. On the other hand t_μ is the vector that coincides in Σ with the time part of $S_{\mu\nu}$:

$$t_k^{(\Sigma)} = iS_{k4}^{(\Sigma)} , \qquad t_4^{(\Sigma)} = 0 . \tag{18}$$

They are space-like, satisfying

$$s_\mu v_\mu = t_\mu v_\mu = 0 . \tag{19}$$

Quite similarly, $S_{\mu\nu}$ is also equivalent to another pair of vectors defined by

$$M_\mu = \tilde{S}_{\mu\nu}p_\nu , \tag{20}$$

$$N_\mu = S_{\mu\nu}p_\nu , \tag{21}$$

which coincide, respectively, with the space and time parts of $S_{\mu\nu}$ in the Π-frame:

$$M_k^{(\Pi)} = m'S_{ij}^{(\Pi)} , \quad (ijk) \sim (123) ; \qquad N_k^{(\Pi)} = im'S_{k4}^{(\Pi)} . \tag{22}$$

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(6) T. TAKABAYASI: *Séminaire L. de Broglie* (1956).

(7) $(ijk) \sim (1\,2\,3)$ signifies that $(i\,j\,k)$ is a cyclic permutation of $(1\,2\,3)$.

We have the relations

$$M_\mu p_\mu = N_\mu p_\mu = 0 , \tag{23}$$

$$M_\mu^2 - N_\mu^2 = (m'^2/2) S_{\mu\nu}^2 , \tag{24}$$

$$m'^2 S_{\mu\nu} = - i\varepsilon_{\mu\nu\varkappa\lambda} M_\varkappa p_\lambda - N_{[\mu} p_{\nu]} . \tag{25}$$

A remarkable fact is that it follows

$$\dot{M}_\mu = 0 , \qquad i.e., \qquad M_\mu = \text{const. of motion} , \tag{26}$$

as the immediate consequence of (1) and (3). Especially its time component

$$M_0 \equiv M_4/i = \sum_{\text{cyc}} S_{ij} p_k , \tag{27}$$

means the scalar product of the internal angular momentum axial vector $\boldsymbol{S} \equiv (S_{23}, S_{31}, S_{12})$ and $\boldsymbol{p}$, *i.e.*, the *helicity* of the particle motion. Noting (22) equation (26) means that the internal angular momentum components in the Π frame are themselves constants of motion.

13. *Associated Lorentz tetrads.* – A particle with internal angular momentum is always associated with two different orthogonal pairs, $\{s_\mu, v_\mu\}$ and $\{M_\mu, p_\mu\}$. Now each of them can be completed into a Lorentz frame of axes (« tetrad » or « vierbein ») by introducing the following vectors:

$$\begin{cases} A_\mu^{(1)} = p_\mu - m v_\mu - \delta \cdot s_\mu/(s_\varrho^2) , \\ A_\mu^{(2)} = i\varepsilon_{\mu\nu\varkappa\lambda} p_\nu s_\varkappa v_\lambda ; \end{cases} \tag{28}$$

$$\begin{cases} B_\mu^{(1)} = p_\mu - (m'^2/m)\{v_\mu + \delta \cdot M_\mu/M_\varrho^2\} , \\ B_\mu^{(2)} = i\varepsilon_{\mu\nu\varkappa\lambda} p_\nu M_\varkappa v_\lambda , \end{cases} \tag{29}$$

where δ is the pseudo-scalar defined by

$$\delta \equiv \tilde{S}_{\mu\nu} p_\mu v_\nu = s_\mu p_\mu = - M_\mu v_\mu . \tag{30}$$

We can really verify that each set

$$\{A_\mu^{(1)}, A_\mu^{(2)}, s_\mu, v_\mu\} , \tag{31}$$

$$\{B_\mu^{(1)}, B_\mu^{(2)}, M_\mu, p_\mu\} , \tag{32}$$

span a Lorentz frame satisfying the orthogonality conditions. The corres-

ponding unitary vectors are

$$\text{(33)}\qquad \begin{cases} \hat{A}^{(1)}_\mu = A^{(1)}_\mu / K_a\,, \qquad \hat{A}^{(2)}_\mu = A^{(2)}_\mu / (\sqrt{s^2_\varrho}\cdot K_a)\,, \\ K_a \equiv \{(m^2 - m'^2) - \delta^2/s^2_\varrho\}^{\frac{1}{2}}\,, \\ a^3_\mu = s_\mu/\sqrt{s^2_\varrho}\,, \quad a^4_\mu = -iv_\mu\,, \end{cases}$$

and

$$\text{(34)}\qquad \begin{cases} \hat{B}^{(1)}_\mu = (m/m')B^{(1)}_\mu / K_b\,, \qquad \hat{B}^{(2)}_\mu = B^{(2)}_\mu / (\sqrt{M^2_\varrho}\cdot K_b)\,, \\ K_b \equiv \{(m^2 - m'^2) - m'^2\,\delta^2/M^2_\varrho\}^{\frac{1}{2}}\,, \\ b^3_\mu = M_\mu/\sqrt{M^2_\varrho}\,, \qquad b^4_\mu = -i\hat{p}_\mu = -ip_\mu/m'\,. \end{cases}$$

Then each of

$$\text{(31')}\qquad \{\hat{A}^{(1)}_\mu,\ \hat{A}^{(2)}_\mu,\ a^3_\mu,\ a^4_\mu\}$$

and

$$\text{(32')}\qquad \{\hat{B}^{(1)}_\mu,\ \hat{B}^{(2)}_\mu,\ b^3_\mu,\ b^4_\mu\}\,,$$

satisfies the orthonormality conditions of a Lorentz system of axes.

The B-frame (32) consists of two constant axes M_μ and p_μ, and the other two axes $B^{(1)}_\mu$ and $B^{(2)}_\mu$ which rotate in the constant plane normal to the formers.

On the other hand the A-frame (31) precesses around the B-frame (s_μ around M_μ, and v_μ around p_μ, etc.) as we shall see later.

1·4. *Subsidiary conditions.* – The relations stated in Sections **1**·1 to **1**·3 are quite general for a particle with internal angular momentum, as they refer to the conservation laws and their direct consequences. They are, however, not sufficient to define the particle motion, because they consist essentially of the equations of motion for p_μ and $S_{\mu\nu}$, (1) and (3), for the set of variables,

$$\text{(35)}\qquad \{v_\mu,\ p_\mu,\ S_{\mu\nu}\}\,.$$

To complete the system of equations we need to have either the equation of motion for v_μ directly or else certain subsidiary conditions, and in any cases they represent more specific relations.

Examples of subsidiary conditions with physically simple meanings are

(I) colinearity between p_μ and v_μ,

(II) $\qquad t_\mu \equiv S_{\mu\nu}v_\nu = 0\,, \qquad i.e., \qquad S^{(\Sigma)}_{k4} = 0\,,$

(III) $\qquad N_\mu \equiv S_{\mu\nu}p_\nu = 0\,, \qquad i.e., \qquad S^{(II)}_{k4} = 0\,.$

The condition (I) leads to (11a) and (12).

The condition (II) is the one adopted in the theory of Weyssenhoff. It is known that the equations (1), (2), (3) and (II) really form a consistent set for the variables (35) and can be solved exactly.

The condition (III) is more stringent than (II), since (1), (3) and (III) result in both (I) and (II). Thus, it represents the simplest case, which was treated, *e.g.*, by NAKANO [8].

Now, all of these conditions (I), (II) and (III) refer to the variables (35) only, being irrelevant to kinematical variables to describe the internal rotation of the particle. Nevertheless the motion of the particle is determined under any of these conditions. This means that in those cases the internal degrees of freedom of the particle are represented by the *dynamical* variables $S_{\mu\nu}$ alone, while the kinematical variables of the internal rotation are left arbitrary. But we just want to develop a theory in which the internal rotational variables will come into play. We shall find that the theory which we state in Section 2 corresponds to replacing Weyssenhoff's condition (II) by a more general one.

At this point we only remark that such a generalization is conceivable because the condition (II) is not necessarily a unique possibility from physical grounds. (II) is imposed to assimilate the internal dynamical variables of the particle to the conventional concept of the internal angular momentum of a non-relativistic rigid body; there is *a priori* no general reason to require it in so far as we are seeking a purely relativistic theory for a microscopic particle [9]. By abandonning the condition (I) we have already introduced an unconventional property which does not occur in the non-relativistic theory of a free particle.

Finally we note the important condition which is weaker and of much wider applicability than (I) or (II), that is

$$\text{(IV)} \qquad \delta \equiv \tilde{S}_{\mu\nu} p_\mu v_\nu = 0 \,, \qquad i.e., \qquad \boldsymbol{S}^{(II)} \boldsymbol{V}^{(II)} = 0 \,,$$

although this does not represent a sufficient condition to make the whole set of equations a self-contained one.

(IV) is equivalent to

$$\text{(IV')} \qquad s_\mu \dot{t}_\mu = 0 \,,$$

because (3) leads to

$$s_\mu (\dot{t}_\mu + p_\mu) = 0 \,,$$

in general.

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[8] T. NAKANO: *Progr. Theor. Phys.*, **15**, 333 (1956).

[9] This was emphasized, from a somewhat different point of view, by D. BOHM and J. P. VIGIER: *Phys. Rev.*, **109**, 1882 (1958); see also F. HALBWACHS, P. HILLION and J. P. VIGIER: *Nuovo Cimento*, **10**, 817 (1958).

By virtue of (IV), the variables of A- and B-tetrads (28), (29), (33) and (34) acquire the simpler forms:

$$(36)\qquad \begin{cases} A_\mu^{(1)} = p_\mu - mv_\mu\,, & \hat{A}_\mu^{(1)} = A_\mu^{(1)}/(m\beta)\,, \\ A_\mu^{(2)} = (i/m\beta)\varepsilon_{\mu\nu\varkappa\lambda}p_\nu a_\varkappa^3 v_\lambda\,, & \\ B_\mu^{(1)} = p_\mu - (m'^2/m)v_\mu\,, & \hat{B}_\mu^{(1)} = B_\mu^{(1)}/(m'\beta)\,, \\ B_\mu^{(2)} = (i/m\beta)\varepsilon_{\mu\nu\varkappa\lambda}p_\nu b_\varkappa^3 v_\lambda\,, & \end{cases}$$

and we have the relations,

$$A_\mu^{(1)}B_\mu^{(2)} = A_\mu^{(2)}B_\mu^{(1)} = 0\,, \qquad A_\mu^{(1)}b_\mu^3 = a_\mu^3 B_\mu^{(1)} = 0\,,$$

and

$$\dot{N}_\mu = -mB_\mu^{(1)}\,.$$

1·5. *Internal orientation and internal angular velocity.* – Now we need to consider how the internal rotational orientation of a relativistic particle can be described. Obviously it is represented by three orthogonal 3-dimensional vectors $a_k^{(\Sigma)r}$ $(r=1\,2\,3)$ which are attached to the particle, *in the rest frame of the particle* Σ, satisfying

$$a_k^{(\Sigma)r}a_k^{(\Sigma)s} = \delta_{rs}\,. \qquad (r,s=1\,2\,3).$$

We can now define three space-like 4-vectors a_μ^r $(r=1\,2\,3)$ by its property that it becomes in the frame Σ

$$(37)\qquad a_\mu^{(\Sigma)r} = \{a_k^{(\Sigma)r},\,0\}\,.$$

On the other hand the 4-velocity v_μ of the particle becomes $v_\mu^{(\Sigma)} = (0,\,0,\,0,\,i)$ in the rest frame. Hence a_μ^r and $a_\mu^4 \equiv -iv_\mu$ together form a set of four 4-vectors spanning a Lorentz tetrad, satisfying

$$(38)\qquad a_\mu^\xi a_\mu^\eta = \delta_{\xi\eta}\,, \qquad (\xi,\eta=1\,2\,3)$$
$$(\det\{a_\mu^\xi\} = +1)\,.$$

Then this tetrad $\{a_\mu^\xi\}$ unifies all kinematical variables of the particle rotator, representing the translational velocity by its fourth axis, and the internal rotational orientation by its three space axes a_μ^r.

In Section 1·3 we have shown that two different tetrads (31) and (32) can be associated to the particle using the dynamical quantities (35) only. They must not be considered as internally attached to the particle. We distinguish

the kinematical variables $\{a_\mu^\xi\}$ from them. In the special case of colinearity, former tetrads become indefinite since we then have $A_\mu^{(1)} = A_\mu^{(2)} = 0$, $B_\mu^{(1)} = B_\mu^{(2)} = 0$.

From a_μ^ξ we derive the « angular velocity tensor » of the particle by

$$\Omega_{\mu\nu} = \dot{a}_\mu^\xi a_\nu^\xi = -\Omega_{\nu\mu}\,, \tag{39}$$

which is written also in the form

$$\dot{a}_\mu^\xi = \Omega_{\mu\nu} a_\nu^\xi\,. \tag{40}$$

The $\Omega_{\mu\nu}$ is mathematically equivalent to the pair of space like vectors

$$\omega_\mu \equiv \tilde{\Omega}_{\mu\nu} v_\nu\,, \qquad (\omega_\mu v_\mu = 0) \tag{41}$$

and

$$\Omega_{\mu\nu} v_\nu = i\,\dot{a}_\mu^4 = \dot{v}_\mu\,, \tag{42}$$

as we have

$$\Omega_{\mu\nu} = -\,i\varepsilon_{\mu\nu\varkappa\lambda}\omega_\varkappa v_\lambda - \dot{v}_{[\mu} v_{\nu]}\,. \tag{43}$$

The ω_μ becomes in the rest frame

$$\begin{cases} \omega_k^{(\Sigma)} = \Omega_{ij}^{(\Sigma)} = \left[\dfrac{\mathrm{d}a_i^r}{\mathrm{d}t}\cdot a_j^r\right]^{(\Sigma)}, & (i\,j\,k) \sim (1\,2\,3), \\ (\omega_4^{(\Sigma)} = 0)\,, \end{cases} \tag{44}$$

which is nothing but the 3-dimensional angular velocity axial vector ([10]). So we call ω_μ the *covariant angular velocity pseudovector*. Equations (41) and (42) show that $\Omega_{\mu\nu}$ unifies the angular velocity of the internal rotation and the translational acceleration.

Now the subsidiary condition which has simple physical meaning and will lead to the determination of the internal rotation is evidently

$$s_\mu = I\omega_\mu\,, \qquad (i.e.,\quad \tilde{S}_{\mu\nu} v_\nu = I\tilde{\Omega}_{\mu\nu} v_\nu), \tag{V}$$

meaning that in the rest frame the internal angular momentum $S_{ij}^{(\Sigma)}$ and internal angular velocity $\Omega_{ij}^{(\Sigma)}$ be parallel with a single constant coefficient, the moment of inertia I; namely that the particle be « rigid » and spherically symmetric.

We would like, however, to construct a theory with a different approach, and afterwards we shall find that the condition (V) is really satisfied in certain special cases of our theory.

([10]) Cfr. T. TAKABAYASI: *Nuovo Cimento*, **7**, 118 (1958).

By the way, it is sometimes convenient to consider physical quantities in the intertial frame which momentarily coincides with the tetrad $\{a_\mu^\xi\}$ itself. This is a special rest frame of the particle and so we denote it as Σ^0. For instance, the components of $\Omega_{\mu\nu}$ viewed in Σ^0 are given by

$$\Omega^{[\xi\eta]} = a_\mu^\xi a_\nu^\eta \Omega_{\mu\nu} = a_\mu^\xi \dot{a}_\mu^\eta = -\Omega^{[\eta\xi]} . \tag{45}$$

Also we shall find it useful to introduce the vectors Φ_μ and Ψ_μ which coincide respectively with the space and time parts of $\Omega_{\mu\nu}$ in the frame Π, such that

$$\Phi_k^{(\Pi)} = m' \Omega_{ij}^{(\Pi)} \quad (i\,j\,k) \sim (1\,2\,3) , \qquad \Psi_k^{(\Pi)} = im' \Omega_{k4}^{(\Pi)} . \tag{46}$$

They are

$$\Phi_\mu = \tilde{\Omega}_{\mu\nu} p_\nu , \qquad \Psi_\mu = \Omega_{\mu\nu} p_\nu , \tag{47}$$

and satisfy $\Phi_\mu p_\mu = \Psi_\mu p_\mu = 0$.

1·6. *Spinor representation.* – The kinematical state of a particle rotator has been represented by a Lorentz tetrad $\{a_\mu^\xi\}$. On the other hand it is known that a 4-component spinor ζ just involves those variables. Thus the kinematical state of the rotator is describable in a most compact way with a spinor ζ. This is defined along a world line, $\zeta(\tau)$, and is restricted by the normalization condition

$$(\bar{\zeta}\zeta)^2 + (i\bar{\zeta}\gamma_5\zeta)^2 = 1 , \tag{48}$$

which is natural for ζ to be the description of a single particle [11].

From ζ we can first derive the unitary real 4-vector $i\bar{\zeta}\gamma_\mu\zeta$ which is time-like, $(i\bar{\zeta}\gamma_\mu\zeta)^2 = -1$ with $\bar{\zeta}\gamma_4\zeta > 0$, and hence we can identify it as the particle 4-velocity:

$$\dot{x}_\mu \equiv v_\mu = ia_\mu^4 = i\bar{\zeta}\gamma_\mu\zeta . \tag{49}$$

Second, the real pseudovector $i\bar{\zeta}\gamma_5\gamma_\mu\zeta$ satisfies

$$(i\bar{\zeta}\gamma_5\gamma_\mu\zeta)^2 = 1 \quad \text{and} \quad (i\bar{\zeta}\gamma_5\gamma_\mu\zeta)(i\bar{\zeta}\gamma_\mu\zeta) = 0 ,$$

so we identify it as the third axis of the tetrad:

$$a_\mu^3 = i\bar{\zeta}\gamma_5\gamma_\mu\zeta . \tag{50}$$

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[11] This is observed by recalling that for the Dirac field $\psi(\boldsymbol{x}, t)$, $\{(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5\psi)^2\}^{\frac{1}{2}}$ corresponds to the density of the number of particles in the local rest frame. (See T. TAKABAYASI: *Suppl. Progr. Theor. Phys.*, no. 4 (1957)).

Defining further a_μ^1 and a_μ^2 by

$$a_\mu^1 + i a_\mu^2 = \bar{\zeta}^c \gamma_\mu \zeta \,, \tag{51}$$

where ζ^c denotes the « charge conjugate » $\zeta^c = C\bar{\zeta}^T$, with

$$C^\dagger C = 1 \,, \qquad C^T = -C \,, \qquad \gamma_\mu^T = -C^{-1}\gamma_\mu C \,,$$

we obtain exactly the Lorentz tetrad $\{a_\mu^\xi\}$ associated to the spinor, satisfying (38) ([12]).

The spinor ζ restricted by (48) involves, besides $\{a_\mu^\xi\}$, one more (pseudoscalar) variable χ, which is related to ζ by ([12])

$$\bar{\zeta}\zeta = \cos\chi \,, \qquad i\bar{\zeta}\gamma_5\zeta = \sin\chi \,. \tag{52}$$

This variable χ may look redundant from a simply kinematical point of view, but it will play a characteristic role in the dynamics of the rotator.

1·7. *Equation of motion and the structure of the internal angular momentum.* – After clarifying the fact that ζ works just as the representation of the kinematical variables of a particle rotator, we are now in a position to postulate an equation of motion directly for ζ. Then this immediately implies a covariant theory of particle rotator defining the time evolution of the internal rotational orientation as well as of the translational motion. In this approach we need not consider separately the form of the subsidiary condition mentioned in Section **1**·4. The general dynamical relations stated in Sections **1**·1 to **1**·3 must also follow from our ζ-formalism, in so far as the theory is Lorentz covariant, where the physical quantities will in general be derivable from ζ as its various bilinear covariants.

The essential problem is therefore to find out a reasonable form of the equation of motion for ζ. As we want to have a theory which would eventually lead to elementary particle models, we take the analogy to the Dirac equation, requiring for the equation of motion of ζ to be a differential equation of the first order with respect to proper time. However, for the equation to be automatically consistent with the normalization condition (48), it must contain a self-interaction term of the third order in ζ.

The explicit form of our equation is given by (71) in Section **2**, but from the general character of that equation of motion follow not only the general relations of Section **1**·1 to **1**·3, but also the fact that the internal angular

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([12]) T. Takabayasi: *Nuovo Cimento*, **7**, 118 (1958); *Compt. Rend. Ac. Sci.*, **246**, 64 (1958); F. Gürsey: *Nuovo Cimento*, **5**, 784 (1957).

momentum of the particle must be represented by

$$S_{\mu\nu} = -(i/2)Q(\bar{\zeta}\gamma_{[\mu}\gamma_{\nu]}\zeta)\,, \tag{53}$$

where Q means the second structure constant of the particle. But the expression (53) can be reduced to one in terms of the variables of (49), (50) and (52), so that

$$S_{\mu\nu} = Q(i\cos\chi\cdot\varepsilon_{\mu\nu\varkappa\lambda}v_{\varkappa}a^3_{\lambda} - \sin\chi\cdot v_{[\mu}a^3_{\nu]}) = Q(\cos\chi\cdot a^1_{[\mu}a^2_{\nu]} + i\sin\chi\cdot\varepsilon_{\mu\nu\varkappa\lambda}a^1_{\varkappa}a^2_{\lambda})\,. \tag{54}$$

This gives

$$s_{\mu} = \quad Q\cos\chi\cdot a^3_{\mu}\,, \tag{55}$$

$$t_{\mu} = -Q\sin\chi\cdot a^3_{\mu}\,. \tag{56}$$

The former shows that the unitary spin pseudovector coincides with a^3_{μ}: $s_{\mu}/\sqrt{s^2_{\varrho}} = a^3_{\mu}$, while the latter indicates that our $S_{\mu\nu}$ indeed does not satisfy the condition (II), except in the special case $\chi = 0$: instead it satisfies the condition:

$$\frac{t_{\mu}}{s_{\mu}} = -\operatorname{tg}\chi\,, \qquad i.e., \qquad \frac{S^{(\Sigma)}_{k4}}{S^{(\Sigma)}_{ij}} = i\operatorname{tg}\chi\,, \qquad (i\,j\,k)\sim(1\,2\,3)\,, \tag{VI}$$

meaning that the two vectors s_{μ} and t_{μ} are colinear ([13]). (VI) also indicates the rôle of the additional variable χ implied in ζ.

On the other hand the condition (IV) is satisfied quite generally, and is now written as

$$a^3_{\mu}p_{\mu} = 0\,. \tag{IV*}$$

This must be equivalent to the condition

$$\dot{\chi} = 0\,, \qquad i.e., \quad \chi = \text{constant of motion}, \tag{IV'*}$$

because (IV') is now written as (IV'*).

Assuming (IV*), the expression (54) gives the helicity vector in the form

$$M_{\mu} = Qm(\cos\chi\cdot a^3_{\mu} - \beta\sin\chi\cdot\hat{A}^{(2)}_{\mu})\,, \tag{57}$$

which must also be constant of motion. We have also $M^2_{\mu} = Q^2m^2\xi^2$, with

$$\xi \equiv \left(1 - \frac{m'^2}{m^2}\sin^2\chi\right)^{\frac{1}{2}}. \tag{58}$$

([13]) The generalization of Weyssenhoff's condition (II) to the form (VI) was suggested by D. Bohm and J. P. Vigier (see ref. ([9])).

Thus

$$b_\mu = M_\mu/(Qm\xi)\,. \tag{59}$$

The components of $S_{\mu\nu}$ in the special rest frame Σ^0 are particularly simple:

$$\begin{cases} S^{[12]} = Q\ \cos\chi\,, \qquad S^{[34]} = iQ\sin\chi\,, \\ S^{[13]} = S^{[14]} = S^{[23]} = S^{[24]} = 0\,. \end{cases} \tag{60}$$

The scalar relations

$$\begin{cases} S^2_{\mu\nu} = 2Q^2\cos 2\chi\,, \qquad S_{\mu\nu}\widetilde{S}_{\mu\nu} = 2Q^2\sin 2\chi\,, \\ Q^2 = s^2_\mu + t^2_\mu = \sum_{cyc} (S^{(\Sigma)}_{ij})^2 + (S^{(\Sigma)}_{k0})^2 \end{cases} \tag{61}$$

are derived most easily by employing (60).

As we shall see our theory will really correspond to the generalization of the theory of Weyssenhoff by replacing the condition (II) by (VI), with the condition (IV'*). If (VI) is given, the structure of $S_{\mu\nu}$ is determined as in the form (54).

2. – Equations of motion.

2·1. *Lagrangian and basic equations.* – According to the argument given in Section **1** the Lagrangian for our particle in the free case must be of the form:

$$\begin{aligned} \mathcal{L}[\bar\zeta, \zeta, x_\mu, p_\mu, \Lambda] = &- iQ(\bar\zeta\dot\zeta - \dot{\bar\zeta}\zeta) + p_\mu(\dot x_\mu - i\bar\zeta\gamma_\mu\zeta) + \\ &+ (\Lambda/2)\{(\bar\zeta\zeta)^2 + (i\bar\zeta\gamma_5\zeta)^2 - 1\}\,, \end{aligned} \tag{62}$$

where p_μ and Λ are Lagrangian multipliers to ensure the relation (49) and the constraint (48).

First the variation of $\mathcal{L}$ with respect to x_μ gives (1). Since p_μ is the quantity canonically conjugate to x_μ, it really signifies the particle momentum-energy vector.

Next the variation regarding $\bar\zeta$ gives

$$2iQ\dot\zeta - ip_\mu\gamma_\mu\zeta + \Lambda\{(\bar\zeta\zeta)\zeta - (\bar\zeta\gamma_5\zeta)\gamma_5\zeta\} = 0\,, \tag{63}$$

which is the basic equation of our theory [14].

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[14] An equation of motion rather similar to the present one was proposed by PROCA: *Journ. Phys. et Rad.*, **15**, 65 (1954); **17**, 81 (1956); *Nuovo Cimento*, **2**, 962 (1955).

Obviously this equation is analogous to Heisenberg's non-linear equation ([15]), meaning that our theory may be regarded as the particle analogue to Heisenberg's field theory.

The conjugate of (63) is

$$-2iQ\dot{\bar{\zeta}} - ip_\mu\bar{\zeta}\gamma_\mu + \Lambda\{(\bar{\zeta}\zeta)\bar{\zeta} - (\bar{\zeta}\gamma_5\zeta)\bar{\zeta}\gamma_5\} = 0\,. \tag{64}$$

2·2. *Equations for physical quantities.* – We shall now deduce the equations for the physical quantities of the particle, such as v_μ and $S_{\mu\nu}$, from the spinor equation. The time evolution of a certain bilinear quantity $\bar{\zeta}O\zeta$, where O represents a Dirac matrix, is derived from (63) and (64) in the form

$$2iQ\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\zeta}O\zeta) = -ip_\lambda(\bar{\zeta}[\gamma_\lambda, O]\zeta) - \Lambda(\bar{\zeta}\gamma_5\zeta)(\bar{\zeta}[\gamma_5, O]\zeta)\,. \tag{65}$$

First, putting $O = 1$ in (65) we get $\mathrm{d}(\bar{\zeta}\zeta)/\mathrm{d}\tau = 0$, namely $\bar{\zeta}\zeta$ is a constant of motion [and $i\bar{\zeta}\gamma_5\zeta$ is also due to (48)]. Thus we obtained the condition (IV'*), as expected. On the other hand inserting $O = \gamma_5$ into (65) we obtain the condition (IV*), as it should.

Now we put $O = \gamma_\mu$ in (65), resulting in

$$Q\dot{v}_\mu = -Q^{-1}S_{\mu\nu}p_\nu + \Lambda\sin\chi\cdot a^3_\mu\,, \tag{66}$$

where $S_{\mu\nu}$ is the expression (53). Equ. (66), together with (1), (IV*), leads to $p_\mu v_\mu = \mathrm{const}$, which we are to identify as (5).

Next we insert $O = \gamma_\mu\gamma_5$ into (65), to obtain

$$Q\dot{a}^3_\mu = \sin\chi\cdot(p_\mu + \Lambda v_\mu)\,. \tag{67}$$

His equation is

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = p_\mu\gamma_\mu\xi/\{(\bar{\xi}\xi)^2 + (i\bar{\xi}\gamma_5\xi)^2\}^{\frac{1}{2}}\,,$$

where the spinor ξ is not subject to any constraint. F. Gürsey: *Nuovo Cimento*, **5**, 784 (1957) investigated the problem from a geometrical point of view in a general way. D. Bohm, F. Halbwachs, G. Lochak and J. P. Vigier: *Séminaire L. de Broglie* (1956), proposed another equation,

$$(\bar{\xi}\xi - \bar{\xi}\gamma_5\xi\cdot\gamma_5)\dot{\xi} + (\bar{\xi}\dot{\xi} - \bar{\xi}\gamma_5\dot{\xi}\cdot\gamma_5)\xi = 0\,.$$

([15]) W. Heisenberg: preprint (1958). Note that in the *c*-number theory, such as we are treating here, Heisenberg's non-linear term $(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma_\mu\gamma_5\psi$ equals $-\{(\bar{\psi}\psi)\psi - (\bar{\psi}\gamma_5\psi)\gamma_5\psi\}$.

This entails, with the aid of (1), (IV*), (6) and (5), that

$$\sin\chi\cdot(m'^2_{\mu\nu}+\Lambda m)=0\,. \tag{68}$$

Assuming $\chi\neq 0$ ([16]) this gives

$$\Lambda=-m'^2/m\,. \tag{69}$$

Inserting this, (67) is now written

$$Q\dot{a}^3_\mu=\sin\chi\cdot\left(p_\mu-\frac{m'^2}{m}v_\mu\right)\equiv\sin\chi\cdot B^{(1)}_\mu, \tag{70}$$

while the original equation (63) is expressed as

$$2iQ\dot{\zeta}-ip_\mu\gamma_\mu\zeta-(m'^2/m)\{(\bar{\zeta}\zeta)\zeta-(\bar{\zeta}\gamma_5\zeta)\gamma_5\zeta\}=0\,, \tag{71}$$

which can also be written

$$2iQ\dot{\zeta}-ip_\mu\gamma_\mu\zeta-(m'^2/m)\exp[i\chi\gamma_5]\cdot\zeta=0\,, \tag{71'}$$

where χ is related to ζ by (52) but it is a constant of motion.

Finally we put $O=\frac{1}{2}\gamma_{[\mu}\gamma_{\nu]}$ in (65) and get exactly (3). This ascertains that the expression (53) is really the internal angular momentum tensor of our particle.

We now reexpress (66), with the aid of (69), (54) and (IV*), to obtain

$$Q\dot{v}_\mu=-i\cos\chi\cdot\varepsilon_{\mu\nu\varkappa\lambda}v_\varkappa a^3_\lambda p_\nu+m\beta^2\sin\chi\cdot a^3_\mu=m\beta(-\cos\chi\cdot\hat{A}^{(2)}_\mu+\beta\sin\chi\cdot a^3_\mu)\,. \tag{72}$$

We have reached thus the self-contained simultaneous equations of motion for a^3_μ and v_μ, (70) and (72), which do not involve a^1_μ and a^2_μ at all, and they can be solved exactly (see Section 2·4).

For the special case $\chi=0$, they are simplified to

$$\dot{a}^3_\mu=0\,,\qquad \dot{v}_\mu=-(i/Q)\varepsilon_{\mu\nu\varkappa\lambda}v_\varkappa a^3_\lambda p_\nu\,, \tag{73}$$

respectively. This demonstrates that our theory really reduces to that of Weyssenhoff for $\chi=0$, since as we have proved elsewhere ([17]) the latter theory is equivalently represented by the set of equations (1), (IV*), and (73). For

([16]) For the case $\chi=0$, see the remark following the equation (73).

([17]) See ref. ([6]).

$\chi = 0$, however, equ. (68) was trivial and (69) did not follow, allowing Λ to be arbitrary—in general a function of τ. Nevertheless the case $\chi = 0$ is included in (71) as the limiting case $\chi \to 0$. This situation implies that the Weyssenhoff theory is described in terms of a spinor ζ restricted by the conditions ([18])

$$\bar{\zeta}\zeta = 1\,, \qquad \bar{\zeta}\gamma_5\zeta = 0\,, \tag{74}$$

and obeying the equation of motion

$$2iQ\dot{\zeta} - ip_\mu\gamma_\mu\zeta + \Lambda\zeta = 0\,, \tag{75}$$

in which Λ can be an arbitrary function of τ, and this arbitrariness does not affect the physical result following from it.

2·3. *Internal angular velocity.* – Before entering the problem of solving {(70), (72)}, we shall consider the equation of motion for a_μ^1 and a_μ^2. We note that from (71) it follows

$$2iQ\dot{\zeta}^c - ip_\mu\gamma_\mu\zeta^c - (m'^2/m)\{(\bar{\zeta}^c\zeta^c)\zeta^c - (\bar{\zeta}^c\gamma_5\zeta^c)\gamma_5\zeta^c\} = 0\,, \tag{76}$$

where we have taken into account that

$$\bar{\zeta}^c\zeta^c = -\bar{\zeta}\zeta\,, \qquad \bar{\zeta}^c\gamma_5\zeta^c = -\bar{\zeta}\gamma_5\zeta\,.$$

Equ. (76), together with the relation

$$(\bar{\zeta}^c\zeta^c)^2 + (i\bar{\zeta}^c\gamma_5\zeta^c)^2 = 1\,,$$

indicates that our basic equations are invariant under « charge conjugation ». Using (71) and (76) we can derive the time evolution of $\bar{\xi}^c O\zeta$, *i.e.*, those of a_μ^1 and a_μ^2.

However, we shall obtain, through a simpler procedure, the relation which is implied in the original (71) but not included in the equations regarding the variables r_μ, a_μ^3 and $S_{\mu\nu}$ derived in Section 2·2. First we note that, if a_μ^3 and a_μ^4 are given, $\{a_\mu^1, a_\mu^2\}$ imply only a single independent variable in view of the condition (38). That is the angle specifying the rotational orientation of the a_μ^1 (or a_μ^2) axis in the plane normal to both a_μ^3 and a_μ^4. Thus the remaining equation must be a single relation which determines the angular velocity of the a_μ^1

([18]) Such ζ is nothing but the relativistic Cayley-Klein parameter.

(or a_μ^2) axis in that plane, which is represented by the component

$$\Omega^{[12]} = \dot{a}_\nu^4 a_\nu^2 = -\dot{a}_\nu^1 a_\nu^2 . \tag{77}$$

The above fact is also clear from the following analytical considerations. Suppose the equation of motion for a_μ^1 is written as

$$\dot{a}_\mu^1 = f_\mu . \tag{78}$$

This is equivalent to three scalar equations

$$a_\mu^\xi \dot{a}_\mu^1 = -a_\mu^1 \dot{a}_\mu^\xi = -\Omega^{[1\xi]} = a_\mu^\xi f_\mu , \qquad (\xi = 2, 3, 4) \tag{79}$$

of which the two, with $\xi = 3$ and 4, are derived from (70) and (72) as

$$\Omega^{[13]} = (\sin\chi/Q) p_\mu a_\mu^1 , \qquad \Omega^{[14]} = i(\cos\chi/Q) p_\mu a_\mu^2 . \tag{80}$$

Thus the only new relation contained in (78) is

$$\Omega^{[12]} = -a_\mu^2 f_\mu , \tag{81}$$

which is just to give the expression for (77).

It remains now to find out the explicit form of (81). We multiply (71) by $\bar{\zeta}\gamma_5$ on the left and utilize (IV*), then we get

$$\bar{\zeta}\gamma_5\bar{\zeta} = \dot{\bar{\zeta}}\gamma_5\zeta = 0 . \tag{82}$$

On the other hand we have the kinematical identity [19],

$$\bar{\zeta}\gamma_5\dot{\zeta} - \dot{\bar{\zeta}}\gamma_5\zeta = \sin\chi \cdot a_\varrho^1 \dot{a}_\varrho^2 - i\cos\chi \cdot a_\varrho^3 \dot{a}_\varrho^4 . \tag{83}$$

Equations (82) and (83) yield

$$\Omega^{[34]}/\Omega^{[12]} = -i\,\mathrm{tg}\,\chi . \tag{84}$$

Noting

$$\Omega^{[34]} = a_\varrho^3 \dot{a}_\varrho^4 = -i(m\beta^2/Q)\sin\chi , \tag{85}$$

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[19] The identities to reduce $\bar{\zeta}O\dot{\zeta} - \dot{\bar{\zeta}}O\zeta$ to expressions in the variables (a_μ^ξ, χ) were given in T. TAKABAYASI: *Nucl. Phys.*, **6**, 477 (1958).

we reach the relation looked for:

$$\Omega^{[12]} = (m\beta^2/Q)\cos\chi\,, \tag{86}$$

which is a constant of motion.

In the case of Weyssenhoff $\Omega^{[12]}$ becomes arbitrary, but we may still regard this case as the limit $\chi \to 0$ in our theory, and then the definite value, $m\beta^2/Q$, is assigned to $\Omega^{[12]}$.

The complete form of the internal angular velocity tensor is

$$\Omega_{\mu\nu} = Q^{-1}\{i\cos\chi\cdot\varepsilon_{\mu\nu\varkappa\lambda}B^{(1)}_{\varkappa}a^{3}_{\lambda} + \sin\chi\cdot B^{(1)}_{[\mu}a^{3}_{\nu]}\}\,, \tag{87}$$

which is derived by use of (86), (80) and similar equations (87) is equivalent to the set of all equations of motion for a^{ξ}_{μ} in view of (40). The angular velocity pseudovector is derived from (87) in the form

$$\omega_{\mu} = \frac{m\beta}{Q}\left(\frac{\beta}{Q}s_{\mu} + \sin\chi\cdot\hat{A}^{(2)}_{\mu}\right). \tag{88}$$

Thus in the Weyssenhoff case ($\chi = 0$) the condition (V) is exactly satisfied, with the moment of inertia

$$I = Q^2/(m\beta^2)\,. \tag{89}$$

On the other hand we have

$$\Phi_{\mu} = (m'^2\beta/Q)\sin\chi\cdot\hat{A}^{(2)}_{\mu}\,, \tag{90}$$

$$\Psi_{\mu} = -(m'^2\beta/Q)\cos\chi\cdot\hat{A}^{(2)}_{\mu}\,, \tag{91}$$

whence

$$\Psi_{\mu}/\Phi_{\mu} = -\operatorname{ctg}\chi\,, \quad i.e., \quad \Omega^{(\Pi)}_{k4}/\Omega^{(\Pi)}_{ij} = i\operatorname{ctg}\chi\,, \qquad (i\,j\,k)\sim(1\,2\,3). \tag{92}$$

Equ. (90) indicates that in the Weyssenhoff case we have

$$\Omega^{(\Pi)}_{ij} = 0\,, \tag{93}$$

namely, the particle does not rotate viewed in the Π-frame.

In the limit of $m' \to m$, the moment of inertia (89) diverges unless we assume $Q \to 0$ at the same time. Really in the case of colinearity (I), we have

$$\Omega_{\mu\nu} = 0\,, \qquad a^{\xi}_{\mu} = \text{const} \qquad (\xi = 1\,2\,3\,4). \tag{94}$$

Namely the particle does not rotate at all, whereas the $S_{\mu\nu}$ are constants generally non-vanishing.

2·4. *Solution of the equation of motion.* – From the analysis in the preceding subsections it has become clear that the equations of motion for our particle are represented essentially by (70) and (72) for $\{a_\mu^3, v_\mu\}$. Instead of solving them directly, however, it is simpler first to determine the motion of the B-frame (32), in which the third and fourth axes, M_μ and p_μ, are already constant. Now the equation (72) is transformed into the following equation of motion for $B_\mu^{(1)}$:

$$\dot{B}_\mu^{(1)} = -(i/mQ^2)\varepsilon_{\mu\nu\varkappa\lambda} M_\varkappa p_\lambda B_\nu^{(1)} = -(m'\xi/Q)\varepsilon_{\mu\nu\varkappa\lambda} b_\varkappa^3 b_\lambda^4 B_\nu^{(1)} . \tag{95}$$

This indicates at once that $B_\mu^{(1)}$ rotates in the plane normal both to $b_\mu^3 = M_\mu/(mQ\xi)$ and $ib_\mu^4 = \hat{p}_\mu$ with the angular velocity

$$\omega = m'\xi/Q . \tag{96}$$

Hence $B_\mu^{(2)}$ also rotates in a similar manner. Explicitly we have the solution

$$B_\mu^{(1)} = m'\beta(b_\mu^1 \sin\omega\tau + b_\mu^2 \cos\omega\tau) , \tag{97}$$

where b_μ^1 and b_μ^2 denote two constant space-like vectors normal to b_μ^3 and b_μ^4, satisfying

$$b_\mu^\xi b_\mu^\eta = \delta_{\xi\eta} , \qquad (\xi, \eta = 1\,2\,3\,4) \tag{98}$$

$$\det\{b_\mu^\xi\} = +1.$$

This $\{b_\mu^\xi\}$ represents a constant « mean rest frame » of the particle which we have denoted as Π.

From (97) we immediately get

$$v_\mu = -\frac{m}{m'}\beta\,(b_\mu^1 \sin\omega\tau + b_\mu^2 \cos\omega\tau) + \frac{m}{m'^2} p_\mu , \tag{99}$$

while a_μ^3 is obtained from (97) with the use of (70), and taking account of the conditions (IV*) and $(a_\mu^3)^2 = 1$; namely

$$a_\mu^3 = \sin\gamma\cdot(b_\mu^1 \cos\omega\tau - b_\mu^2 \sin\omega\tau) + \cos\gamma\cdot b_\mu^3 , \tag{100}$$

where γ denotes a constant defined by

$$\operatorname{tg}\gamma = \beta \operatorname{tg}\chi .$$

The above result shows that our particle moves in such a way that the velocity v_μ precesses around the constant momentum vector p_μ with the in-

variant circular frequency (96) (*i.e.*, the frequency in the frame Σ), while a_μ^3 precesses around the constant helicity pseudovector M_μ with the same frequency. Therefore when v_μ comes back to the initial value after one period of motion $\Delta\tau = 2\pi/\omega$, a_μ^3 also re-establishes its initial orientation.

On the other hand it can be proved that a_μ^1 (or a_μ^2) involves different periodical components and does not come back to its initial orientation after $\Delta\tau$.

In the special case $\chi = 0$, the oscillatory part of a_μ^3 disappears and we have $a_\mu^3 = b_\mu^3 = \text{const}$, while the expression (99) for v_μ remains unchanged, where the invariant circular frequency is now $\omega_{\text{W}} = m'/Q$. This is exactly the motion known for Weyssenhoff's particle.

We remark here that in the above procedure we have assumed $m' = 0$; if $m' = 0$, the equations of motion have no solution. Thus m' cannot exactly be equal to zero even though it may be infinitesimally small.

To see the motion of our particle slightly more intuitively it is convenient to consider it in the Π-frame, in which $b_\mu^{4(\Pi)} = i\delta_{\mu 4}$, while $\{b_k^{\tau(\Pi)}\}$ spans a constant 3-dimensional orthogonal frame:

$$b_k^{r(\Pi)} b_k^{s(\Pi)} = \delta_{rs} \qquad (r, s = 1\,2\,3)\,, \qquad b_4^{r(\Pi)} = 0.$$

We then see that the 3-dimensional velocity $V^{(\Pi)}$ rotates with the circular frequency

$$\omega^{(\Pi)} = \omega(1 - V^{(\Pi)2})^{\frac{1}{2}} = m'^2\xi/(mQ)\,,$$

and that the particle moves on a circle in a plane parallel to $(\boldsymbol{b}^{1(\Pi)}, \boldsymbol{b}^{2(\Pi)})$-plane with radius $R = m\beta Q/(m'^2\xi)$.

3. – Concluding remarks.

Our first object was to clarify the general aspect of any covariant theory of a particle with internal rotational degrees of freedom. After doing that we introduced a certain precise form for the equation of motion to present a concrete example of theory. Various properties possessed by such a particle related to its internal motion were analysed. Thus far the theory is essentially an extension of the classical theory of relativistic particles. Starting from this wider classical model we need to investigate in the next step whether by its quantization we may reach a plausible model of elementary particles.

To proceed to quantum theory it is reasonable, by the analogy with the case of the Dirac field, to set anti-commutation relations for ζ:

$$\{\zeta_\alpha, \bar{\zeta}_\beta\} = \frac{\hbar}{2Q}\,\delta_{\alpha\beta}\,, \qquad \{\zeta_\alpha, \zeta_\beta\} = \{\bar{\zeta}_\alpha, \bar{\zeta}_\beta\} = 0\,. \tag{101}$$

We can then verify that the components of the internal angular momentum

$$S_k \equiv S_{ij} \overset{\text{def}}{=} -iQ(\bar{\zeta}\gamma_i\gamma_j\zeta)\,, \qquad (i\,j\,k)\sim(1\,2\,3) \tag{102}$$

satisfy the usual commutation relations between angular momentum components:

$$[S_i, S_j] = i\hbar S_k\,, \qquad (i\,j\,k)\sim(1\,2\,3).$$

Accordingly $\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2$ takes eigenvalues $j(j+1)\hbar^2$ with $j = 0, \frac{1}{2}, 1, \ldots$, including half integer spin states ([20]). A consequent treatment of the quantized theory and the interpretation of its results will be studied in a subsequent paper.

What we would like to remark here is that our equation of motion (71) is not necessarily the unique possibility; there is another wide possibility for the particle rotator. For instance we may conceive the equation of motion for ζ which involves ζ^c explicitly. Then the particular decoupling between $\{a_\mu^1\, a_\mu^2\}$ and $\{a_\mu^3, v_\mu\}$ in our theory will no longer be valid. On the other hand it should also be significant to formulate a covariant theory with internal rotation on the basis of a 2-component spinor $\varphi(\tau)$ in place of the 4-component $\zeta(\tau)$, corresponding to a polarized particle.

* * *

The author would like to express his sincere thanks to Professors L. DE BROGLIE and J. P. VIGIER for kind hospitality and exciting discussions at the Institut Henri Poincaré. He is also indebted to Professor D. BOHM for valuable discussions which he had in the autumn 1956.

([20]) This is because our S is defined by (101) and (102), and is algebraically not the same with the orbital angular momentum components $L_k = x_{[i}p_{j]}$ with $[x_i, p_j] = i\hbar\delta_{ij}$.

RIASSUNTO (*)

Si analizzano gli aspetti generali cinematici e dinamici di qualsiasi teoria covariante delle particelle dotate di velocità angolare interna assieme a impulso angolare interno. Le teorie si classificano secondo la forma delle condizioni sussidiarie richieste.

(*) *Traduzione a cura della Redazione.*

Si costruisce una teoria esatta con l'ausilio di uno spinore ζ quale rappresentazione cinematica di tale struttura, postulando la seguente equazione di moto:

$$2iQ\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} - ip_\mu\gamma_\mu\zeta - (m'^2/m)(\bar{\zeta}\zeta)\zeta - (\bar{\zeta}\gamma_t\zeta)\gamma_5\zeta = 0\,,$$

dove ζ è definito lungo una linea d'universo, Q ed m rappresentano le costanti di struttura della particella originale, mentre m' è la massa a riposo osservabile che può assumere un determinato valore costante tra zero ed m. L'equazione contiene un termine di autointerazione del terzo ordine, e si può pertanto ritenere che la teoria rappresenti per la particella l'analogo del campo lineare di Heisenberg. Ciononpertanto è esattamente risolvibile, e il risultato dimostra che la particella, oltre al suo moto rettilineo medio rappresentato dal vettore d'impulso costante p_μ, una zitterbewegung orbitale intorno allo stesso con una precessione sincronizzata del terzo asse interno intorno al pseudovettore di elicità costante. La teoria di Weyssenhoff è compresa nella presente teoria come caso degenerato nel quale la velocità angolare interna diventa fisicamente arbitraria. Si spera che, partendo da questo modello classico generalizzato, si possa ottenere per quantizzazione un modello unificato delle particelle elementari.

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On the Viscosity of Fluids.

A. Carrelli and E. Ragozzino

Istituto di Fisica Sperimentale dell'Università - Napoli

(ricevuto il 30 Marzo 1959)

Summary. — In a fluid with dissymmetric molecules there is a dependence of the viscosity on the gradient of speed. We give here a theory of this effect, which can be developed exactly only for a gas, and we describe experiences made with some liquids and solutions, which, confirming the existence of the variation we have mentioned, show that for some fluids it can become so remarkable as to allow an experimental determination of the characteristic geometrical parameters of the molecules.

It is known that in several cases the particular shape and size of some molecules or macromolecules can influence the viscosity of solutions or dispersions. It is possible, in these cases, that the value of the viscosity depends on the value of the gradient of speed. If the particle has a special shape, for instance a very elongated shape, we can consider in it a particular axis. Besides we can consider a fluid which consists of particles of this kind and we can suppose in it a particular condition of movement, in order to measure the coefficient of viscosity.

Obviously the particle can assume in time, on account of the thermal agitation, any orientation in the fluid; but, if it is in a layer having a speed V, it tends to assume such an orientation as to offer the least obstacle to the flow. This means that, if it is immersed in a thread having a speed V, the particle is submitted to an orientation moment M which is equal to 0 when the molecule or the granule is in that particular position for which it offers the least resistence. The value of this moment, therefore, is 0 when the axis of symmetry of the particle and the vector V are parallel; then, if we name θ the angle this axis and this vector form, we may write, bearing in mind we

must consider M proportional to the speed:

$$M = -KV \sin\theta .$$

K is a coefficient which depends on the dissymetry and therefore on the shape and size of the particle. On account of the moment M, the particle experiences a motion of rotation; we may consider its speed of rotation $\mathrm{d}\theta/\mathrm{d}t$ proportional to M, so that we may write:

$$M = \xi \frac{\mathrm{d}\theta}{\mathrm{d}t} .$$

The problem in which we are interested is like the one, already studied by DEBYE, concerning a dipole submitted to an electric field, and we can say that, while the axes of symmetry of the particles can, on account of the thermal agitation, assume any orientation, the speed of the layer, in which they are, tends to orient them according to the speed: the statistical distribution is not the one foreseen by BOLTZMANN but it is modified, and the calculations, performed in a way similar to the ones concerning dipoles, show that the distribution law f is approximately given by the following relation:

$$f = A\left[1 + \frac{KV}{kT} \exp\left[-\frac{2kT}{\xi}\right] \cos\theta\right] . \tag{1}$$

Consequently, in the layer having the speed V, the average value $\bar{\Delta}$ of the molecular diameter, according to the direction of V, is expressed by the following formula, in which f is given by (1) and has the exponential function equal to 1, because we refer to stationary conditions:

$$\bar{\Delta} = \frac{\int f \cdot \Delta \cdot \cos\theta \, \mathrm{d}\Omega}{\int f \cdot \mathrm{d}\Omega} .$$

If we consider particles whose diameter has a projection equal to $\Delta_0 + \Delta_1 \cos\theta$, according to the direction of V (that is to say particles consisting of a cylinder having length Δ_1 and, at the extremities, two half spheres with diameter Δ_0), we shall have in this case:

$$\bar{\Delta} = \frac{\int_0^{\pi} f(\Delta_0 + \Delta_1 \cos\theta) \, \mathrm{d}\Omega}{\int_0^{\pi/2} f \, \mathrm{d}\Omega} = \Delta_0 + \frac{2\pi \cdot A \cdot \Delta_1 \int_0^{\pi/2} [1 + (KV/kT) \cos\theta] \cos\theta \cdot \sin\theta \, \mathrm{d}\theta}{2\pi \cdot A \int_0^{\pi/2} [1 + (KV/kT) \cos\theta] \sin\theta \, \mathrm{d}\theta} ,$$

and, if KV/kT is constant, we shall easily obtain:

$$\bar{\Delta} = \Delta_0 + \frac{\Delta_1/2 + \Delta_1 KV/3kT}{1 + KV/2kT}. \tag{2}$$

The average value Δ of the diameter, therefore, depends on the speed V of the layer.

We shall obtain the same result if we consider particles having a different shape, for instance a parallelepipedic or ellipsoidal shape.

For the particles having parallelepipedic shape, the projection of the diameter, according to V, is given by the following expression:

$$\Delta = \Delta_0 \sin\theta + (\Delta_0 + \Delta_1)\cos\theta ,$$

where Δ_0 is the width and Δ_1 the length of the particle.

The average value of Δ is:

$$\bar{\Delta} = \frac{\Delta_0(\pi/4) + \Delta_0 KV/3kT + (\Delta_0 + \Delta_1)/2 + (\Delta_0 + \Delta_1)(KV/3kT)}{1 + KV/2kT}.$$

For the particles having ellipsoidal shape, the values of Δ and $\bar{\Delta}$ are:

$$\Delta = 2\sqrt{\Delta_0^2 \sin^2\theta + \Delta_1^2 \cos^2\theta}; \quad \begin{cases} 2\Delta_0 = \text{minor axis}, \\ 2\Delta_1 = \text{major axis}, \end{cases}$$

$$\bar{\Delta} = \Delta_1 + \frac{\Delta_0^2}{\sqrt{\Delta_1^2 - \Delta_0^2}} \log \frac{\Delta_1 + \sqrt{\Delta_1^2 - \Delta_0^2}}{\Delta_0} + \frac{2KV}{3kT} \frac{\Delta_1^3 - \Delta_0^3}{\Delta_1^2 - \Delta_0^2}.$$

In any case, therefore, $\bar{\Delta}$ depends on V.

If we simplify the problem and refer to a fluid such as a gas, the coefficient of internal friction, for the layer in which the average value of diameter is $\bar{\Delta}$, is given by the following expression:

$$\bar{\eta} = \frac{1}{3\sqrt{2}\pi} \frac{m\bar{c}}{\bar{\Delta}^2}.$$

Viscosity, therefore, becomes a function of V.

We must notice that:

1) For a gas, the theoretically foreseen percent variation of η, at least with the particles we have previously considered, the value of V passing from 0 to ∞, is rather limited (about 30%).

2) As soon as we are able to determine experimentally the variation law of η as function of V, we can find the characteristic values Δ_0, Δ_1, of the particles of the gas.

For the liquids we cannot give a theoretical expression to the coefficient of viscosity, but we imagine that also for them the coefficient η may depend on the speed with a law similar to the one obtained for a gas. Hence it is not possible to ascertain theoretically the dependence of η on the speed in the layer, but it is possible to realize in a fluid, such as a solution or a dispersion, a gradient of speed which is constant in respect of time but for every measurement has a different value.

The law which expresses the dependence of η on V can thus be experimentally determined.

To this purpose we made some measurements of the coefficient of viscosity of fluids at first by means of a Searle's viscosimeter and then by means of an Otswald's viscosimeter.

In the first of these instruments the moment, caused by the uniform rotation of a cylinder, containing the tested substance and revolving round its axis with a frequency ν which was able to assume values from 0.1 to 24 Hz, was balanced by the moment caused by the torsion of a wire or a spring joined to an inner cylinder coaxial to the first cylinder. The measurements of the values of ν lower than 5 Hz were made directly while the greater frequencies were measured by using a collar, having a thin iron plate, joined to the external cylinder, and a coil placed in front of the collar, at some millimeters distance from the latter. The coil was connected to a circuit governing the vertical deflections of an oscilloscope on which we were able to read directly the values of ν. We used the Searle's viscosimeter with torsion wire in order to examine more accurately the behaviour of the tested substances with values of ν not greater than 3 Hz.

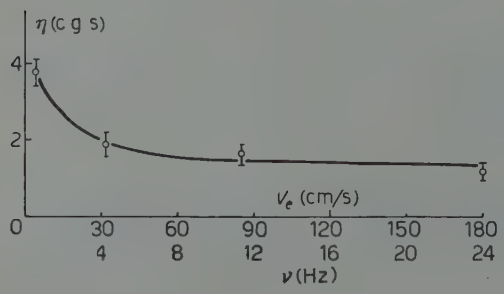

Fig. 1.

In the Otswald's viscosimeter the tested substances flew through a little pipe, 60 cm long and 0.44 cm wide, under pressure differences Δ_p variable from 30 to 250 mm Hg. A pressure regulator permitted to have a constant Δ_p during every measurement.

In any case, by means of the two instruments, it was possible to measure η with various gradients of speed.

The fluids, whose coefficient of viscosity was measured by means of these

special viscosimeters are: glycerine, para in benzol, tylose in water, nylon in formic acid, polyvinyl acetate in acetone, aniline resin in butyl alcohol.

Figs. 1-7, show some results we obtained by means of the Searle's viscosi-

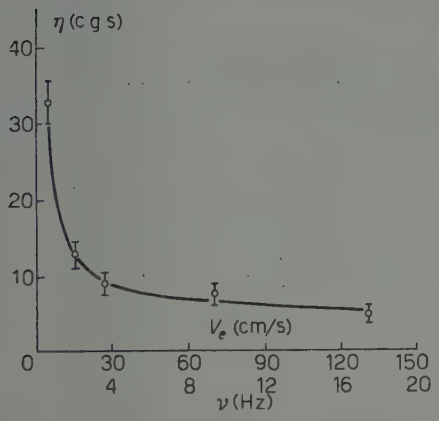

Fig. 2.

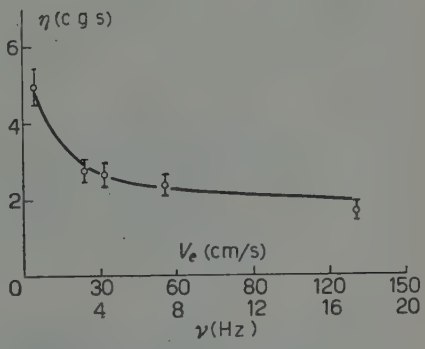

Fig. 3.

meter with spring (*): Figs. 1, 2 refer to the ones obtained with two dispersions of para in benzol, Figs. 3, 4, to the ones obtained with two dispersions of tylose in water and the others refer respectively to glicerine, a solution of

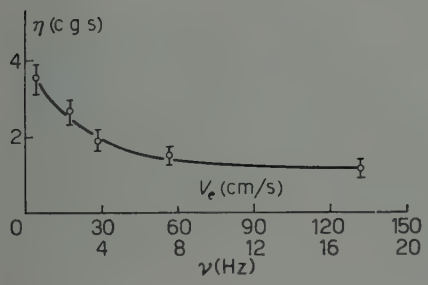

Fig. 4.

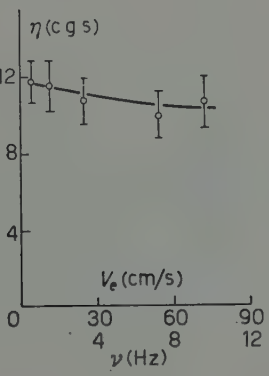

Fig. 5.

aniline resin in butyl alcohol and a solution of polyvinyl acetate in acetone Figs. 8, 9 refer to results we obtained by means of the Searle's viscosimeter with torsion wire respectively with a solution of smoked para in benzol and

(*) The values of η are plotted also against the velocity V_e of the external cylinder.

a dispersion of tylose in water. Finally Figs. 10, 11 (*), show the results obtained by means of the Ostwald's viscosimeter and still referring to a solution of smoked para in benzol and dispersion of tylose in water.

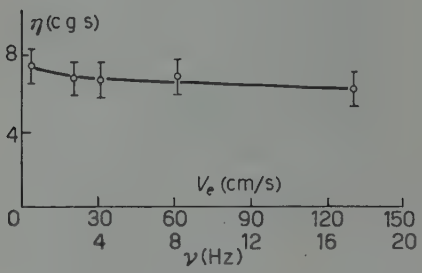

Fig. 6.

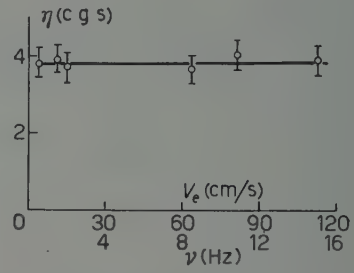

Fig. 7.

All these results prove at first that there are pure substances or solutions for which the dependence of η on the gradient of speed, on account of the molecular orientation, lacks, while, on the contrary, there are other substances which show a dependence of η either on the frequency (Searle's viscosimeter) or on the speed (Ostwald's viscosimeter).

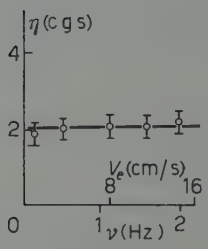

Fig. 8.

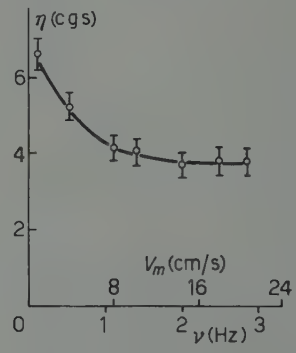

Fig. 9.

Besides we must notice that the preceding variation of η is much greater than that theoretically foreseen for a gas: in some cases the values of η obtained at the highest frequencies are even four times smaller than those obtained at the lowest frequencies.

(**) The values of η are expressed also in terms of the average flow velocity V_m.

Lacking a definite theory on the viscosity of liquids, we must presently be satisfied with the qualitative results coming to the conclusion that this

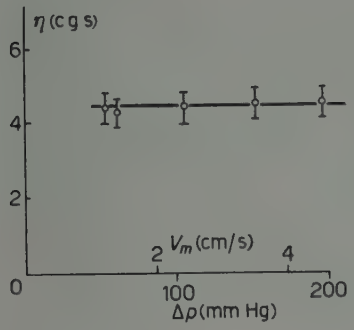

Fig. 10.

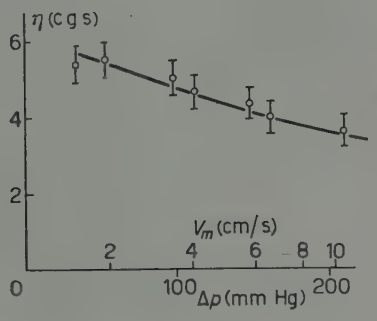

Fig. 11.

dependence of η on V, on account of the molecular orientation, with some substances is very remarkable and, therefore, we have a method suitable to verify experimentally molecular asymmetries.

RIASSUNTO

In un fluido con molecole dissimmetriche esiste una variabilità del coefficiente di attrito interno con il gradiente di velocità. È data una teoria di questo effetto, la quale può naturalmente svilupparsi in modo preciso solo per i gas, e sono descritte esperienze compiute su liquidi puri e soluzioni, che, confermando l'esistenza della predetta variabilità, mostrano che essa è per taluni fluidi molto notevole e tale da consentire una determinazione sperimentale dei parametri geometrici caratteristici delle molecole considerate.

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α-Particles Straggling in Mica and Aluminum.

F. Demichelis

Istituto di Fisica Sperimentale del Politecnico - Torino

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Summary. – The straggling of the α-particles in mica and aluminum is examined. For mica the experimental values agree reasonably well with the theoretical values obtained by applying the Livingston and Bethe formula; for aluminum the experimental values differ greatly from the theoretical ones.

1. – Introduction.

If we were to examine the energy loss distribution which is attained by a beam of high velocity α-particles of the same initial energy after passing through a small thickness of stopping material such that only a small fraction of the energy is lost we should find a statistical fluctuation in the amount of energy lost.

These fluctuation effects which are known as « straggling » have been discussed by Bethe and Ashkin [1], Allison and Warshaw [2], etc.

The first detailed treatments of this phenomenon were given independently by Flamm [3] and by Bohr [4] on the basis of simple considerations of classical mechanics.

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[1] H. Bethe and J. Ashkin: *Experimental Nuclear Physics* (New York, 1953), Chap. 2, p. 166.

[2] S. K. Allison and S. D. Warshaw: *Rev. Mod. Phys.*, **25**, 779 (1953).

[3] L. Flamm: *Akad. Wiss. Wien, Math.-naturw. Kl.*, **123**, 1393 (1914); **124**, 597 (1915).

[4] N. Bohr: *Phil. Mag.*, (6) **30**, 581 (1915).

The energy loss straggling theory has been worked out by LANDAU [5] and improved by BLUNCK and LEISEGANG [6].

Experimental researches on the straggling were made by LEWIS and WYNN-WILLIAMS, BRIGGS, BENNET and recently by PORTER and HOPKINS [7].

Looking apart from the rare cases in which an α-particle suffers a collision with a nucleus so violent that it loses a considerable part of its energy, the contribution of nuclear collisions to the straggling for fast α-particles may, therefore, be entirely neglected.

Then we may affirm that the energy losses $\Delta_\varepsilon E$ in electronic collisions will be distributed according to a simple Gaussian law

$$W(\Delta_\varepsilon E) = \frac{1}{\sqrt{2\pi}\,\Omega_\varepsilon} \exp\left[-\frac{(\Delta_\varepsilon E - \Delta_\varepsilon E_0)^2}{2\Omega_\varepsilon^2}\right],$$

where the mean square deviation is [8]

$$\Omega_\varepsilon^2 = 4\pi N \Delta R z^2 \varepsilon^4 Z, \tag{1}$$

here ε is the charge of the electron,

N the number of atoms per cubic centimeter of the material,

Z the nuclear charge of the atoms of the stopping material,

z the charge number of the incident particle,

ΔR the thickness of stopping material.

The classical formula of Bohr (1) is especially simple being independent from the velocity of the incident α-particles.

By applying the Born approximation as in the theory of stopping, LIVINGSTON and BETHE [9] obtained for the mean square deviation the following expression

$$\Omega_\varepsilon^2 = 4\pi\varepsilon^4 z^2 N \Delta R \left(Z' + \sum_n k_n \frac{I_n Z_n}{mu^2} \log \frac{2mu^2}{I_n}\right), \tag{2}$$

where Z' is the total number of « effective » electrons defined as the number of electrons in the atom, Z, excluding those in the inner shells for which

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[5] L. LANDAU: *Žu. Èksp. Teor. Fiz.*, **8**, 201 (1944).

[6] O. BLUNCK and S. LEISEGANG: *Zeits. f. Phys.*, **128**, 500 (1950).

[7] W. B. LEWIS, C. E. WYNN-WILLIAMS: *Proc. Roy. Soc. (London)*, A **136**, 349 (1932); G. H. BRIGGS: *Proc. Roy. Soc. (London)*, A **114**, 313 (1927); W. E. BENNETT: *Proc. Roy. Soc. (London)*, A **155**, 419 (1936); P. T. PORTER and J. I. HOPKINS: *Phys. Rev.*, **91**, 209 (1953).

[8] N. BOHR: *Det. Kgl. Dan. Videnskab. Selskab.-fys. Meddelelser*, **18**, 8 (1948).

[9] M. S. LIVINGSTON and H. A. BETHE: *Rev. Mod. Phys.*, **9**, 245 (1937).

$I_n > 2mu^2$. The sum goes over the shells which are not excluded, k_n are certain constants between $\frac{2}{3}$ and $\frac{4}{3}$, Z_n is the number of electrons in the n-th shell and I_n their average excitation energy.

For high energies the sum over n may be neglected and the formula (2) becomes the classical formula of Bohr (1).

The straggling effects are of importance whenever a beam of charged particles passes through a thickness of stopping material; particularly they must be taken into account when detectors are used employing thin windows (Geiger counters, NaI(Tl) crystals,...). In fact the fluctuations in the amount of energy lost worsen the resolution of the detector.

Therefore it seemed useful to us to examine the straggling of α-particles in the absorbers usually used as windows: mica and aluminum.

For mica our experimental results are in agreement with the Livingston and Bethe formula (2). We cannot say the same thing for the aluminum.

2. – Measuring technique.

The source of α-particles is $^{210}_{84}$Po ($E = 5.298$ MeV), having an intensity of $\approx 10\ \mu c$, obtained (Harwell) by electrodeposition and covered with a thin mica foil ($1\ mg/(cm)^2$).

Another source of the same α-particles is $^{210}_{82}$Pb. We used also this source which was not covered and consists of homogeneous active deposits, on the backing foil of mica, as a thin layer in order to diminish the influence of auto-absorption.

The detector consists of a CsI(Tl) crystal (thickness ≈ 0.8 mm, diameter ≈ 10 mm) followed by a 6292 Du Mont photomultiplier; the α-particles enter the surface ($\approx 12\ (mm)^2$) of the crystal near normal incidence.

The pulses coming from the photomultiplier are sent, conveniently amplified, to RIDL 100 channel analyzer.

The detector's calibration was obtained with the same α-particles of $^{210}_{84}$Po and the α-particles of $^{214}_{84}$Po ($E = 7.680$ MeV) and of $^{212}_{84}$Po ($E = 8.776$ MeV).

The pulse height is proportional to the energy of the α-particles.

The thickness of the mica foils, interposed between the source and the detector varies from $0.392\ mg/(cm)^2$ to $2.100\ mg/(cm)^2$.

Accurate values for the mica foil thickness were obtained either with a Mettler microbalance (sensitivity: 10^{-3} mg) or with the interferential method of the « cannelures » ([10]). The uniformity of the mica foils was verified by means of a polarization microscope.

The thickness of the aluminum foils varied from $0.148\ mg/(cm)^2$ to

([10]) M. Françon: *Encyclopedia of Physics*: Vol. **24** (Berlin, 1956), p. 171.

2.340 mg/(cm)² and was determined with the microbalance. The aluminum foils are not of uniform thickness. By weighing we get only the « mean thickness » of the foil.

An examination of an aluminum foil having a thickness of 1.49 mg/(cm)² with the electron microscope and by means of the Taylor-Hobson surface analyzer gives the following results. On the surface of the foil, which appears to be the best when examined through the optical microscope, we can observe a great deal of rulings whose depth is of the order of 0.05 μm; the maximum depth observed is 0.13 μm. On the other surface, which appears very irregular upon examination through the optical microscope, the depth of the sinkings is even of ≈ 2 μm. Moreover we can estimate that the number of the holes in 100 (mm)² of an aluminum foil of the thickness of 0.01 mm is $\approx 4 \cdot 10^{-5}$ (mm)².

The distance between the source and the detector is selected in order to accept the approximation of the point source; on the other hand this distance is known and it is possible to deduce the energy of the α-particles entering the surface of the detector. Moreover we have measured the distance between the source and the absorbers in order to deduce the energy of the α-particles entering the absorbers.

The α-particles entering the surface of the detector after passing through the various thin foils of the absorbers, have a residual kinetic energy which is obtained by applying the range-energy relations for α-particles of Bethe and Ashkin ([1]). These relations agree substantially with the curves given by Livingston and Bethe ([9]).

3. – Results.

For the various thicknesses of the absorbers we have obtained the spectrum of the $^{210}_{84}$Po α-particles and we have measured the full width η of peak at $\frac{1}{2}$ maximum.

Fig. 1 shows the experimental results obtained for mica. We may observe the measured full widths η as a function of the thickness of the absorbers.

Fig. 2 shows the experimental results for aluminum.

Fig. 3 shows the resolution η/E of the detector as a function of the thickness of the absorbers; E is the residual kinetic energy of the α-particles after passing through the absorbers.

Scintillation pulses from monoergic particles give lines which are often found to have a gaussian shape; nevertheless there are many factors which can increase the line width ([11]).

([11]) E. BREITENBERGER: *Progress in Nuclear Physics*, Vol. 4, (London, 1955), Chap. 2, p. 56.

The straggling phenomena give a gaussian shape. Without mica or aluminum absorbers we have a gaussian result of the scintillation line and of the straggling effects produced by the air, by the mica foil covering the source and, eventually, by the straggling produced in the same source.

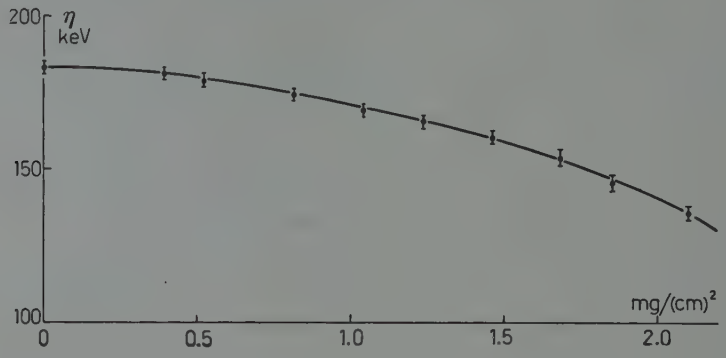


Fig. 1.

By inserting the absorbers between source and detector, the α-particles enter the detector with another kinetic energy and the absorbers produce straggling.

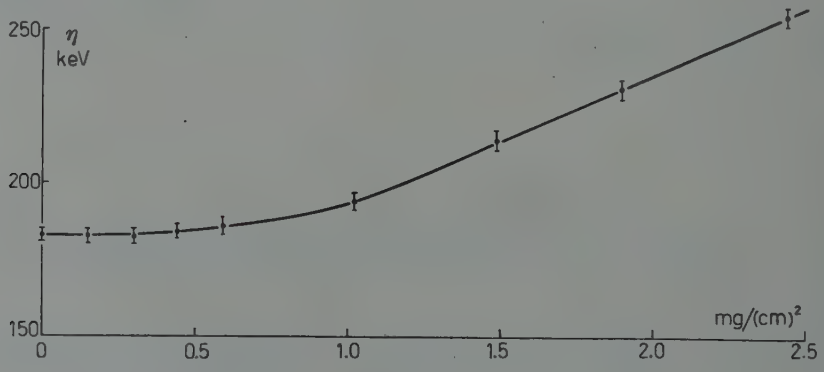


Fig. 2.

According to BASHKIN and co-workers ([12]), the resolution in percent varies inversely as the square root of the particle energy; moreover the particles under observation are not strictly monochromatic but possess an energy spread.

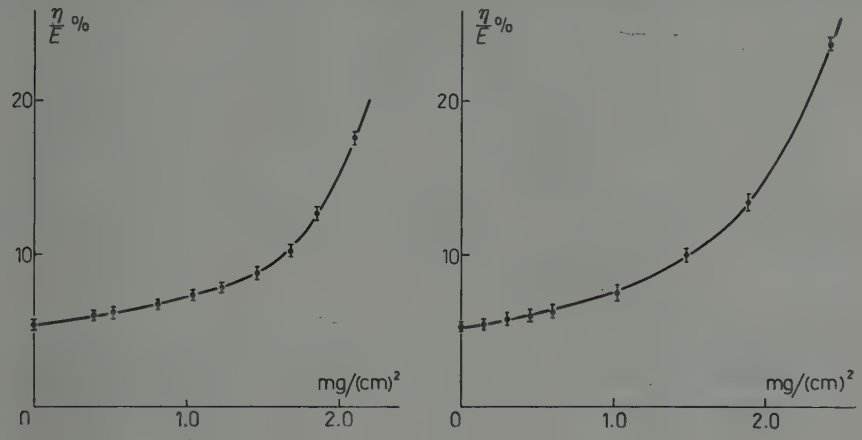
The full width of a gaussian line at $\frac{1}{2}$ maximum being $2\sqrt{2 \cdot \ln 2}$ times

([12]) S. BASHKIN, R. R. CARLSON, R. A. DOUGLAS and J. A. JACOBS: *Phys. Rev.*, **109**, 434 (1958).

the root mean square deviation we have

$$\eta^2 = \eta_{en}^2 + \eta_{air}^2 + \eta_{abs}^2 ,$$

where η_{en}, η_{air}, η_{abs} are the full width relative to scintillation line, to straggling of air (and eventually to straggling of the mica foil on the source; this value remains unaltered in the measurements), and to straggling of the absorbers.



a) mica *b) aluminum*

Fig. 3.

Taking into account the above considerations we can write:

$$\eta^2 = K^2 E\left(1 + \frac{dE}{E}\right) + \eta_{air}^2 + \eta_{abs}^2 , \tag{2}$$

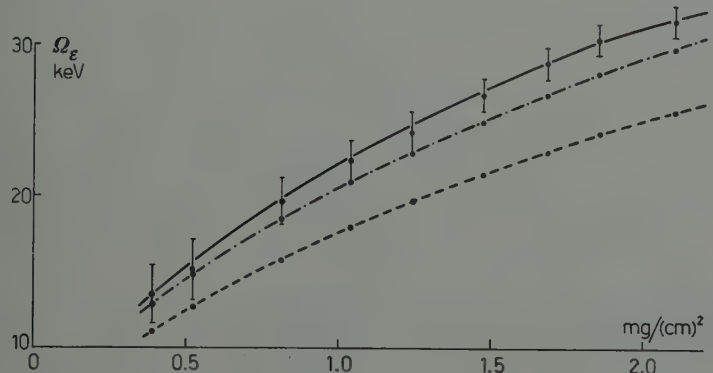


Fig. 4.

here dE is the spread of the energy produced by the straggling η_{air} and η_{abs}.

Without absorbers we can obtain the value of K^2 from the following re-

lation

$$\eta^2 = K^2 E\left(1 + \frac{\mathrm{d}E}{E}\right) + \eta_{\mathrm{air}}^2 ,$$

η_{air} being known according to the Livingston and Bethe expression (2). The estimate of K^2 is provided by the measurements made using α-particles emitted by $^{210}_{84}$Po, $^{212}_{84}$Po, $^{214}_{84}$Po sources.

Inserting the values of K^2 and η_{air}^2 in the above relation we have obtained the values η^2 for the various absorbers.

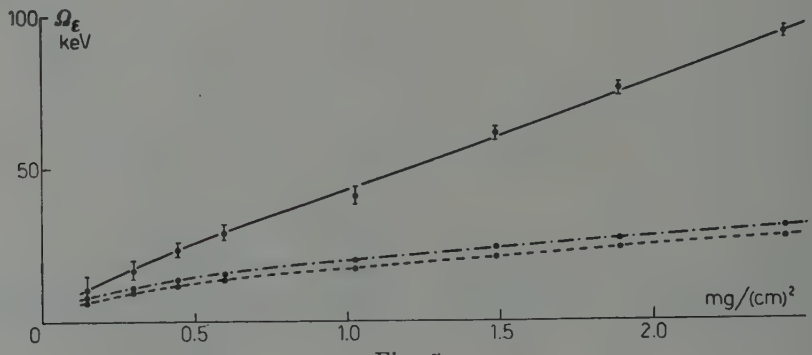


Fig. 5.

The full line of Fig. 4 shows the values so obtained for Ω_ε as a function of the thickness of the absorbers.

The values Ω_ε for aluminum are reported in Fig. 5 (full line).

4. – Discussion.

We shall find it convenient to verify the agreement between the theoretical and the experimental values.

The dashed line of Fig. 4 shows the values of Ω_ε for mica according to the relation of Bohr (1). It is evident that the experimental values differ greatly from Bohr's values and we must calculate the theoretical values according to the Livingston and Bethe formula.

The energy of the α-particles entering the absorbers is 3.55 MeV. The shells for which $I_n > 2mu^2$ must be excluded in the sum of (2).

First of all we must calculate I_K, I_L, ..., for the aluminum and for the components of the mica ($K_2O\,3Al_2O_3\,6SiO_2\,2H_2O$).

The average excitation potential of the K shell is

$$I_K = \lambda_K(\theta_K)\, Z_{K\mathrm{eff}}^2 R_{\mathrm{H}} ,$$

here $Z_{K\text{eff}} = Z - 0.3$ is the effective nuclear charge in the K shell,
R_{H} is the mean ionization potential of the hydrogen atom,
λ_K is the mean ionization potential and
θ_K is the observed energy difference between a K-electron in the ground state and the lowest unoccupied state in units $Z^2_{K\text{eff}} R_{\text{H}}$.

Tipical values of λ_K were calculated by BROWN [13]; by interpolation we can deduce the values for our substances. The values of I_K are reported in Table I.

TABLE I.

	H	O	Al	Si	K
Z	1	8	13	14	19
mu^2	966.70	966.70	966.70	966.70	966.70 V
I_K	7.34	773.30	2080.88	2410.53	4457.81 V
I_L	3.46	52.42	245	325	750 V

We can observe that for Al, Si and K $I_K > 2mu^2$. This means that the K-electrons are ineffective in interactions with the incident particles.

We must then know the contribution of L-electrons. Therefore the values of I_L must be calculated.

WALSKE [14] has investigated the stopping power for L-electrons. From his calculations we can deduce the following relation

$$I_L = \tfrac{1}{4} \lambda_L(\theta_L) Z^2_{L\text{eff}} R_{\text{H}} , \tag{3}$$

here, according to SLATER [15], is $Z_{L\text{eff}} = Z - 4.15$ and λ_L, θ_L have the same meaning than for the K shell, but are related to the L shell.

But λ_L as a function of θ_L is reported for values of $\theta_L < 0.35$ corresponding to $Z = 20$. Moreover WALSKE says that the calculations of the stopping number B_L and the correction term C_L are not reliable for $\theta_L < 0.45$ *i.e.* for $Z < 30$. The atoms which form mica and aluminum have all $Z < 20$.

We have then proceeded in the following manner. For atoms like carbon, nitrogen, oxigen, ... having $Z \leqslant 10$ we have used the relation

$$\log I = \left(1 - \frac{1 + \frac{1}{2}f}{Z}\right) \log I' + \frac{1 + \frac{1}{2}f}{Z} \log I_K ,$$

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[13] L. M. BROWN: *Phys. Rev.*, **79**, 297 (1950).
[14] M. C. WALSKE: *Phys. Rev.*, **88**, 1283 (1952); **101**, 940 (1956).
[15] J. C. SLATER: *Phys. Rev.*, **36**, 57 (1930).

where I' is the average ionization potential of all electrons outside the K shell,

I is the average of the excitation potential over *all* electrons; the values of I are deduced from Bakker and Segrè's experiments ([16]),

$f(\theta_K)$ is the oscillator strength.

From this relation we obtain $I_L = I'$ for $Z \leqslant 10$. For $Z < 20$ we have I_L from Walske's formula (3). The results are reported in Fig. 6; the dashed line is obtained by interpolation. The deduced values of I_L are summarized in Table I.

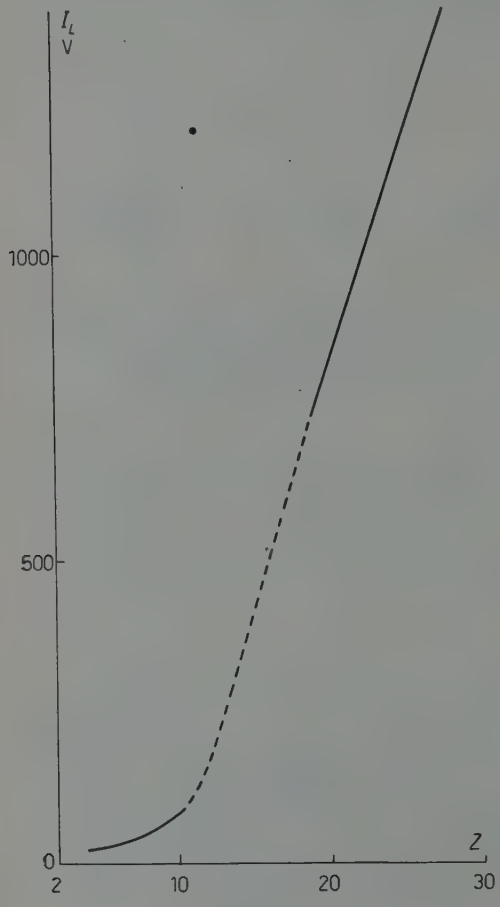

Fig. 6.

We must take into account also the contribution of the M-electrons of the atoms Al, Si, K. However, such calculations have not been carried out as yet for shells other than the L shell. Thus the best that can be done at present is to consider the above correction to Bohr's formula, which appears in the formula (2), less than the effective one.

For mica the contribution of the K-electrons to the sum of formula (2) is 1.038 and the contribution of the L-electrons is 3.017. The expression in brackets assumes the value 12.852. The values of Ω_ε from the formula (2) are reported in Fig. 4 (dotted and dashed line).

For aluminum the only contribution of the L-electrons to the sum of formula (2) is 5.592. The values of Ω_ε are reported in Fig. 5 (dotted and dashed line); the dashed line represents the values according to Bohr's formula.

We may then conclude:

The straggling of $^{210}_{81}$Po α-particles in mica is in agreement with the Livingston and Bethe formula. In aluminum the straggling is not measurable because of the irregularity of the foils.

([16]) C. J. Bakker and E. Segrè: *Phys. Rev.*, **81**, 489 (1951).

Moreover the resolution of the detector, when a thin aluminum foil is used as a window, is not worse than the resolution obtained with a mica foil. Therefore, for what concerns the detector's resolution it is convenient to use thin mica foils, unless thin aluminum foils having a thoroughly uniform thickness can be made.

* * *

The author should like to express sincere thanks to Prof. E. PERUCCA for his kind interest and useful discussions and is particularly grateful to Prof. C. PANSERI, Director of the « Istituto Sperimentale dei Metalli Leggeri » in Milan who allowed the examination of the aluminum foils through the electron microscope and to Dr. M. PAGANELLI for his co-operation in scanning the aluminum foils and in providing all the data on his measurements.

Finally I wish to express my gratitude to Consiglio Nazionale delle Ricerche which through an adequate grant gave me the possibility to dispose of a 100 channel analyzer.

RIASSUNTO

Viene esaminato lo « straggling » delle particelle α negli assorbitori più comunemene usati: mica e alluminio. Mentre i risultati sperimentali relativi alla mica si accordano con i valori ottenuti secondo Livingston e Bethe, quelli relativi all'alluminio se ne scostano sensibilmente.

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Intensity Distribution of High Energy Pions in the Atmosphere.

A. Subramanian and S. D. Verma

Tata Institute of Fundamental Research - Bombay

(ricevuto il 18 Aprile 1959)

Summary. — The vertical intensity distribution of pions in the atmosphere, in the energy range $(10^{10} \div 10^{12})$ eV, is calculated from the energy spectrum of muons at sea level. It is found that, at all depths in the atmosphere, pions should account for only a minor fraction of the intensity of nuclear active particles in the above energy range. A comparison of the energy spectrum of pions at production with the primary nucleonic spectrum leads to the result that the average inelasticity in collisions of nucleons with air nuclei, in the energy range $(10^{11} \div 10^{13})$ eV, lies between 0.5 and 0.3, if reasonable limits to the multiplicity of mesons produced in such collisions are set. It is pointed out that the generation of pions by pions in the atmosphere may have appreciable effect in the zenith angular dependence of muons underground, and if so, this leads to the result that pions are more efficient in producing pions than nucleons, in collisions with air nuclei.

1. – Introduction.

A knowledge of the intensity distribution of high energy ($\geqslant 10$ GeV) pions in the atmosphere, besides leading to a better description of the phenomenology of the cosmic radiation in the atmosphere, is interesting from at least two points of view. One is that it gives the experimentalist the intensities he must encounter in designing experiments to study high energy pion interactions in detail using the cosmic ray beam in the atmosphere. The other is that the intensity of pions in relation to the intensity of nucleons in the atmosphere gives a datum to fix the average inelasticity in the collisions of high energy nucleons with air nuclei [1].

It is well known that one can hope to determine the intensity of pions in

[1] K. Greisen and W. D. Walker: *Phys. Rev.*, **90**, 915 (1953).

the lower atmosphere from measurements of the excess of charged particles among the high energy nuclear active particles. However, it is not possible to interpret this « charge excess » in terms of only pions. Protons could possibly be in excess of neutrons, depending on the interaction and absorption mean free paths of high energy nucleons in air and the charge exchange probabilities in a nuclear collision. Further, a bias could be introduced in the detection of pions and nucleons of the same energy, if the characteristics of the interactions induced by the two types of particles are different. Measurements made so far on the ratio of charged to neutral particles among the high energy nuclear active particles in the lower atmosphere have yielded widely varying values ([1-3]).

An attempt has been made in this paper to resolve the confusing situation regarding the intensity of pions in the atmosphere. This intensity is derived from the sea level energy spectrum of muons, which has been extended up to 10^{12} eV using measurements of the intensity of muons underground. The procedure is the well known one used in the analysis of muon intensities underground ([4-5]). This procedure rests on a few assumptions, the basis for which is discussed in the light of present knowledge and it is shown that a fairly reliable estimate of the vertical intensity of pions in the atmosphere can be obtained. The uncertainties involved in the procedure, as well as the general conclusions which the results of the analysis lead to, are discussed. Finally, we point out that the anomaly reported ([6]) between the zenith angular dependence of muons and their vertical rate of absorption underground can be removed if a fraction of muons underground have their parent pions originating in nuclear collisions of higher energy pions in the atmosphere. This interpretation, if correct, leads to the result that, at corresponding energies, pions are more efficient in producing pions than nucleons in collisions with air nuclei.

2. – Deduction of pion intensity and energy distribution in the atmosphere from the differential energy spectrum of muons at sea level.

Although the procedure adopted here has already been used by several authors ([4,5]) we shall reproduce here the basic assumptions involved and the method of calculation. The following are assumed:

([2]) M. Deutschmann: *Zeits. f. Phys.*, A **9**, 477 (1954).

([3]) L. A. Farrow: *Phys. Rev.*, **107**, 1687 (1957).

([4]) P. H. Barret, M. Bollinger, G. Cocconi, Y. Eisenberg and K. Greisen: *Rev. Mod. Phys.*, **24**, 133 (1952).

([5]) N. L. Grigorov: *Usp. Fiz. Nauk USSR*, **58**, 599 (1956).

([6]) B. V. Sreekantan, S. Naranan and P. V. Ramanamurthy: *Proc. Ind. Acad. Sci.*, **43**, 113 (1956).

a) All muons are generated by the decay of pions only.

b) The interaction mean free path in air for nucleons and pions is the same.

c) The component responsible for the generation of pions is exponentially distributed in the atmosphere with a mean free path independent of energy and depth in the atmosphere.

d) The pions retain the direction of their « generating component ».

Assumption *a*) is justified from recent evidence for the non-copius production of particles other than pions in high energy nuclear interactions ([7,8]). Even if a small fraction of muons arise from the decay of K-mesons, since the lifetime of K-mesons is about 10^{-8} s, which is not very much different from that of pions, the results ensuing from assumption *a*) will not be affected appreciably. Assumption *b*) is based on observation of a near « geometric » mean free path for secondaries of high energy nuclear interactions ([7]). Numerous cosmic ray observations on the absorption of nuclear active particles in the atmosphere suggest assumption *c*) ([3,9]). Further, several authors ([10-12]) have found that the flux of nuclear active particles, determined at some depth in the atmosphere, when extrapolated to the top of the atmosphere, using a mean free path 120 g/cm^2, leads to correct values of the primary flux. If we assume *c*) to be rigorously valid, we imply among other things that pions play a negligible role in the development of the cascade of nuclear active particles in the atmosphere. In particular, we imply that regeneration of pions by pions is negligible. If regeneration of pions by pions is important, a strictly exponential absorption will not be valid for the parents of the pions, since the pion intensity distribution in the atmosphere can be expected to follow a non-exponential distribution, because of density variation in the atmosphere and also because pions are secondaries of nucleons. We cannot assert *a priori* that regeneration of pions by pions will be negligible in the atmosphere, because we do not know the nature of pion interactions with air nuclei. Thus we find that even though the intensity of all nuclear active particles in the atmosphere may be observed to follow approximately an exponential law, there is no basis to assume *c*).

([7]) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

([8]) E. Lohrmann and M. W. Teucher: *Phys. Rev.*, **112**, 587 (1958).

([9]) K. P. Rijhova and L. I. Sarič̆eva: *Journ. Exp. Theor. Phys.*, **1**, 572 (1955).

([10]) L. T. Baradzei, V. I. Rubtsov, I. V. A. Smorodin, M. V. Soloviev and B. V. Talinova: *Journ. Exp. Theor. Phys.*, **6**, 12 (1958).

([11]) G. T. Zatsepin, V. V. Krugovykh, E. A. Murzina and S. I. Nikolskij: *Journ. Exp. Theor. Phys.*, **7**, 207 (1958).

([12]) A. E. Čudakov, N. A. Dobrotin, N. L. Grigorov, G. N. Vernov and G. T. Zatsepin: *Suppl. Nuovo Cimento*, **8**, 737 (1958).

In the course of the present analysis, we shall assume that *c*) is valid and evaluate the vertical intensity distribution of pions in the atmosphere. It will be shown, in Section **4**, that assumption *c*) is largely justified even if pions regenerate pions, provided one is content with estimating the magnitude of the uncertainties in the intensity distribution of pions in the atmosphere derived by assuming *c*). There our method of estimating these uncertainties will be to look upon the generating component of pions as consisting entirely of nucleons and to use for the absorption mean free path for pions a value much larger than their interaction mean free path. In such a case it is obvious that we assume that the nucleonic intensity in the atmosphere is distributed exponentially, with a mean free path coinciding with the observed mean free path for the absorption of all nuclear active particles in the atmosphere. The justification for this assumption as regards the analysis made here for the evaluation of the intensity distribution of pions in the atmosphere is only *a posteriori*. It can be inferred from Section **4** that the absolute intensities of pions as deduced by our method are not very sensitive to slight changes of the absorption mean free path of the nucleons in the atmosphere. These absolute intensities of pions account for only a minor fraction of the intensities of all nuclear active particles even in the lower atmosphere and this result in turn is consistent with our assumption regarding the absorption of nucleons in the atmosphere. As for assumption *d*), we find that it is justified for the energy range with which we are concerned here and is supported by the fact that the transverse momentum of pions at production is about 500 MeV/c ([7,13]).

Assumption *c*) allows one to write the source function of pions in the atmosphere as

$$S_\pi(E, x) = \frac{F_\pi(E)}{L_i} \exp\left[-x/L_a\right],$$

where $S_\pi(E, x)\,\mathrm{d}E\,\mathrm{d}x$ is the number of pions of energy E, in interval $\mathrm{d}E$, produced at a vertical depth x in the atmosphere, in a layer $\mathrm{d}x$. L_a and L_i denote the absorption and interaction mean free paths respectively of the component generating pions in the atmosphere. From the foregoing, we now identify the generating component as consisting of nucleons only. $F_\pi(E)$ can then be termed as the production spectrum of pions generated by the nucleons in the atmosphere. We write the diffusion equation for the pions in the vertical direction in the atmosphere as

$$\frac{\partial \pi(E, x)}{\partial x} = -\pi(E, x)\left[\frac{1}{L_i} + \frac{h}{x}\right] + \frac{F_\pi(E)}{L_i} \exp\left[-x/L_a\right].$$

([13]) Institute for Nuclear Study Reports no. INSJ-7, Tokyo (March 1958).

Here $h = Hm_\pi c/\tau_\pi E$, $H = 6.46 \cdot 10^5$ cm [5], $m_\pi =$ the mass of the pion, $\tau_\pi =$ its mean life-time and $c =$ the velocity of light. If E is measured in GeV, $h = 118/E$.

The solution of the above diffusion equation can be written as

$$\pi(E, x) = F_\pi(E)\, \frac{x}{L_i} \exp[-x/L_a] A(y, h)/h\,,$$

where

$$y = x/L'\,,\quad 1/L' = (1/L_i) - (1/L_a) \qquad \text{and} \qquad A(y, h) = h \int_0^1 t^h \exp[-y(1-t)]\,\mathrm{d}t\,.$$

If $\mu(\bar{E})\,\mathrm{d}\bar{E}$ denotes the vertical differential spectrum of muons at sea level,

$$\mu(\bar{E})\,\mathrm{d}\bar{E} = F_\pi(E)\,\mathrm{d}E h \int_0^1 \frac{t^h \{1 - \exp[(-x_0/L_i)(1 - t + t(L_i/L_a))]\}\,\mathrm{d}t}{1 - t(1 - L_i/L_a)}\,.$$

where $\bar{E} = m_\mu E/m_\pi$ takes into account the energy degradation in the π-μ decay. $x_0 = 1032$ g/cm^2 and since $L_i/L_a \gtrsim 0.5$

$$\mu(\bar{E})\,\mathrm{d}\bar{E} \approx F_\pi(E)\,\mathrm{d}E\, \Omega(h)\,,$$

where

$$\Omega(h) = h \int_0^1 \frac{t^h\,\mathrm{d}t}{(1 - t + t(L_i/L_a))}\,.$$

Thus we can write

$$F_\pi(E)\,\mathrm{d}E = \frac{\mu(\bar{E})\,\mathrm{d}\bar{E}}{\Omega(h)}\,,$$

and

$$\pi(E, x) = \frac{m_\mu}{m_\pi} \frac{\mu(\bar{E})}{\Omega(h)} \frac{x}{L_i} \exp[-x/L_a] \frac{A(y, h)}{h}\,.$$

We have thus deduced both $F_\pi(E)$ and $\pi(E, x)$ from $\mu(\bar{E})$.

3. – Details of the calculation and numerical results on the intensity distribution of pions in the atmosphere.

The function $A(y, h)$ has been computed and plotted in Fig. 1. This function has already been computed by ANNIS [14] for a certain range of values

[14] B. ROSSI: *High Energy Particles* (New York, 1952), p. 492.

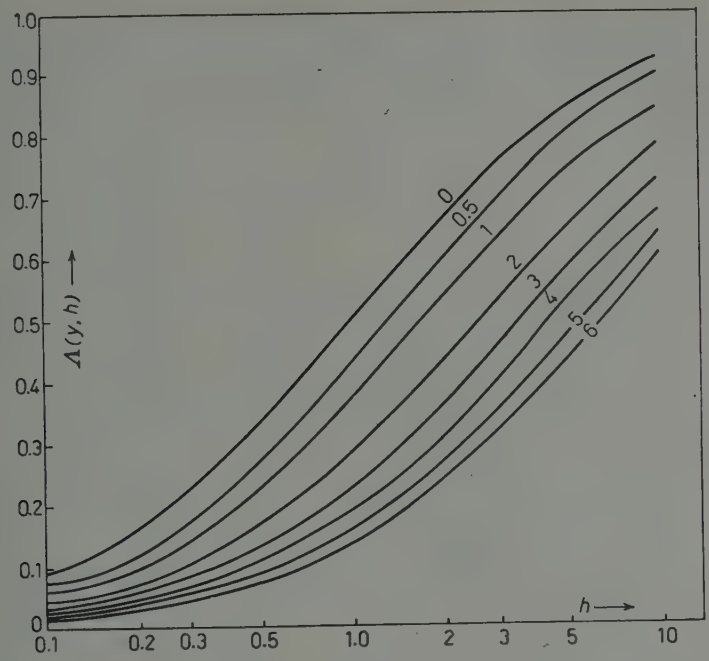

Fig. 1. – The function $\Lambda(y, h)$ plotted against h. The numbers on the curves indicate y values.

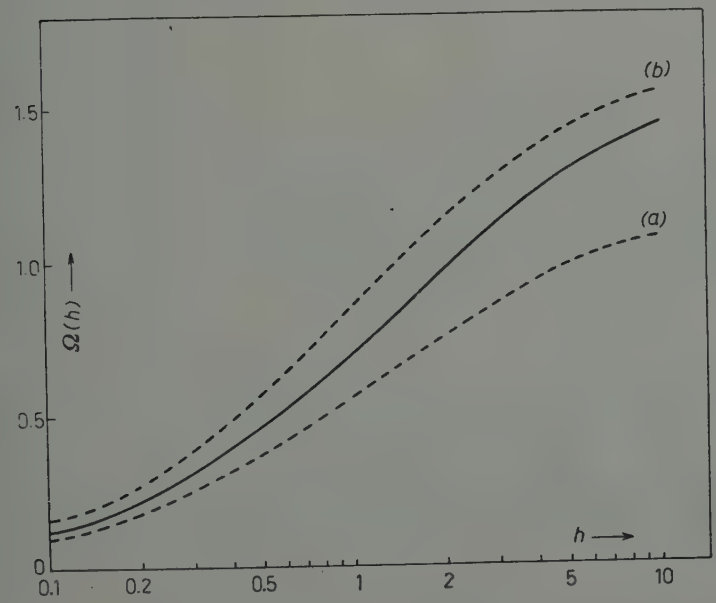

Fig. 2. – Full curve gives the function $\Omega(h)$, as defined in Sect. 2, with $L_i = 70$ g/cm² and $L_a = 120$ g/cm². Curve (*a*) gives $\Omega(h)$ with $L_i = 100$ g/cm² and $L_a = 120$ g/cm². Curve (*b*) gives modified $\Omega(h)$ with $L_\pi = L_a = 120$ g/cm² and $L_i = 70$ g/cm².

of y and h. The function $\Omega(h)$ is given in Fig. 2. We have used $L_i = 70$ g/cm² and $L_a = 120$ g/cm² in computing $\Omega(h)$. The value for L_a is more reliable than that for L_i. However, the intensities of pions will not be affected very much by the possible range of values of L_i from 70 to 100 g/cm², within which L_i is uncertain at the present time ([15]). A discussion of the effect of the uncertainty in L_i follows in the next section. As will be pointed out there, the estimates of inelasticity for collisions of high energy nucleons with air nuclei deduced from the present analysis are consistent with a value of 70 g/cm². The differential muon energy spectrum at sea level is drawn in Fig. 3, using the integral depth intensity curve of muons underground and the energy losses for muons due to various processes ([4]). Thus we have obtained the plot of $F_\pi(E)$ in Fig. 3 and $\pi(E, x)$ in Fig. 4. These have been corrected for the decay of muons in the atmosphere in an approximate manner ([4]).

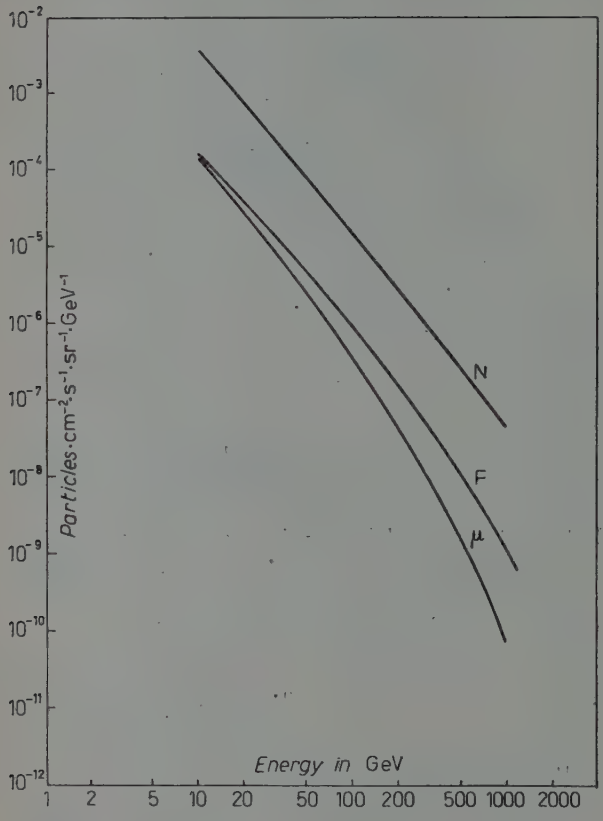

Fig. 3. – Differential energy spectra of different particles: N, nucleons at the top of the atmosphere; F, production spectrum of pions; μ, muons at sea level.

To compare the intensities of pions with those of nucleons in the atmosphere, the integral energy spectrum of all primary nucleons arriving singly or in the form of nuclei was taken to be given by $10\,000/(1+E)^{1.5}$ particles m^{-2} s^{-1} st^{-1} in the range of energies $(10^{10} \div 10^{12})$ eV, where E is measured in GeV. This form of the spectrum has been given by McDonald ([16]) for the latitude sensitive regions of the primary energy. Existing data are not incon-

([15]) A. E. Brenner and R. W. Williams: *Phys. Rev.*, **106**, 1020 (1957).
([16]) F. B. McDonald: *Phys. Rev.*, **109**, 1367 (1958).

sistent with an extension of the above spectrum up to 10^{12} eV ([17]). To obtain the intensity of nuclear active particles at any given depth, an absorption of the primary nucleonic component with a mean free path of 120 g/cm² was used. That this procedure yields correct values of the intensity of nuclear active particles at various depths has been checked by several workers ([10-12]).

Table I gives the ratio of charged to neutral particles among the nuclear active particles, at three depths in the atmosphere, at various energies, on the basis that protons and neutrons are equal in intensities. This should be expected, if we choose L_i to be 70 g/cm² and $L_a = 120$ g/cm² (*). However, it is difficult to be certain about the equality in intensity of protons and neutrons in view of the analysis made by VOROBIEV ([18]) of the charge exchange probabilities in nuclear collisions in air, in the energy range (3 ÷ 10) GeV.

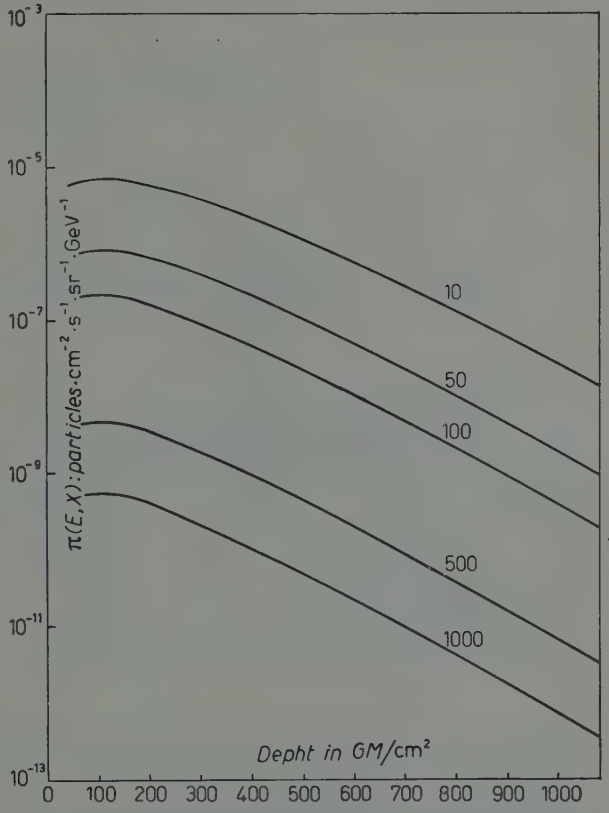

Fig. 4. – Differential intensity distribution of pions in the atmosphere. The numbers on the curves indicate energy of pions in GeV. The pion intensities in the lower atmosphere must be considered uncertain to the extent indicated in Sect. 4.

([17]) B. PETERS: *Lectures on the Origin of Cosmic Radiation* (Varenna, 1958) (mimeographed notes).

(*) Neglecting the presence of neutrons bound in nuclei of the primary cosmic radiation, and assuming a free charge exchange in nuclear collisions of protons and neutrons with air nuclei, a rough formula for the ratio of protons to neutrons at any depth x is given by the expression

$$\frac{\exp[-x/L_a] + \exp[-x/L_i]}{\exp[-x/L_a] - \exp[-x/L_i]}.$$

([18]) V. A. VOROBIEV: *Journ. Exp. Theor. Phys.*, **6**, 203 (1958).

It is clear from Table I that pions are only a small fraction of the nuclear active particles at all energies considered here. This result will hold good for the whole of the atmosphere, since relative intensity of pions will be a maximum at sea

TABLE I. – *Ratio of charged to neutral nuclear active particles at various depths in the atmosphere and at various energies.*

Depth in g/cm²	Energy in GeV				
	10	50	100	500	1000
672	1.05 (1.06)	1.20 (1.37)	1.26 (1.64)	1.24 (1.85)	1.16 (1.45)
840	1.05 (1.07)	1.22 (1.50)	1.26 (1.77)	1.25 (2.22)	1.16 (1.65)
1008	1.06 (1.09)	1.23 (1.57)	1.27 (2.22)	1.25 (2.88)	1.17 (1.12)

level. However, as shown in the next section, the absolute intensities of pions in the lower atmosphere, as derived here, are uncertain because of uncertainties in the magnitude of regeneration of pions by pions. The absolute intensities will also be affected to a certain extent by uncertainties in the values of L_i and L_a. The uncertainties will not be large enough to invalidate the conclusion that at all energies considered here, nucleons form the major part of the nuclear active particles anywhere in the atmosphere. Values in brackets in Table I are those permitted by the uncertainties in the regeneration of pions by pions. Considerations which lead to the evaluation of these figures are outlined in the next section. The analysis made in Section 6 shows that regeneration of pions by pions is likely to be important. In this case, figures in the brackets in Table I should be taken as the more probable values. The effect of regeneration of pions by pions affects most the intensity of pions at the highest energies considered here, (1000 GeV). However, the values of the ratio within brackets at 10 GeV are probably underestimated and could be as large as the values within brackets at 50 GeV (see next Section).

As regards the measured values of the ratio of charged to neutral particles among the nuclear active particles in the atmosphere, a large majority (*) compare well with the values in Table I. A tendency for an increase of the ratio with energy is noted in many experiments ([1,2,15]). This probably indicates that the values within brackets are favoured and if confirmed would mean that regeneration of pions by pions is not negligible. Detailed comparison of the experimental values with the values in Table I is hardly justified at the present time because of the uncertainties in the measured values. It

(*) A list of references for most of the measurements can be found in ([1]). In addition, reference may be made to ([2]), ([3]), ([15]) and ([20]).

is important to obtain accurate values for the charged-to-neutral ratio among the nuclear active particles at various depths in the lower atmosphere so that one can gauge the extent of regeneration of pions by pions in the atmosphere.

4. – Uncertainties involved in the procedure.

To see the extent to which the results obtained by our analysis will be uncertain, we have drawn in Fig. 2, a dashed curve marked (a), which gives $\Omega(h)$ if $L_i = 100$ g/cm². We observe that $\Omega(h)$ decreases by about 25% at $h = 10$ and 17% at $h = 0.1$. These define the limits of uncertainties in $F_\pi(E)$ derived from $\mu(E)$ because of uncertainties in L_i. We have also estimated the uncertainties in $\pi(E, x)$. The uncertainty is largest at sea level since y values are sensitive to changes in L_i. For $L_i = 100$ g/cm², at energies of about 10 GeV, the pion intensity at sea level increases by about 30%, while at energies of about 1000 GeV, the pion intensity increases by a factor of three. The production spectrum of pions as derived here, which is not sensitive to values of L_i, favours independently a value of L_i closer to 70 g/cm² than 100 g/cm². This will be seen from Section 5. Hence we can say that both $F_\pi(E)$ and $\pi(E, x)$, evaluated by using $L_i = 70$ g/cm², should be much less uncertain than indicated above. Analysis similar to the one made above show that both the production spectrum and the pion intensities do not change much by small changes in L_a. Moreover, uncertainties in L_a are much smaller than those in L_i.

We now discuss the effect on the results obtained in the previous Section of pions regenerating pions in the atmosphere. The effect of regeneration o pions will be essentially to make the absorption mean free path of pions through nuclear interactions in the atmosphere effectively larger than L_i. Let us call by L_π this effective absorption mean free path. An approximate relation between L_π and L_i is

$$\frac{L_i}{L_\pi} = 1 - \frac{\varepsilon_\pi^\gamma}{1.5^\gamma n^{\gamma-1}},$$

where ε_π is the average fraction of energy which a pion colliding with an air nucleus loses in producing secondary pions, n the multiplicity of energetic charged pions which, for simplicity, are assumed to share equally the energy going into meson production, γ is the mean exponent in the power law form of the integral energy spectrum of parent pions between the values of the energy of the secondary pions and that of the primary pions, and 1.5 is the factor for taking into account the production of neutral pions. Since γ varies with energy and depth in the atmosphere, L_π will not be a constant. In Table II,

we have given values of γ at selected energies for pions and at different depths in the atmosphere. These values were obtained from the pion intensities deduced earlier. If appreciable regeneration of pions by pions takes place in the atmosphere, the values of the exponents in Table II will be lower estimates

TABLE II. – *Values of the exponents of the integral energy spectra of pions at different depths in the atmosphere and at various energies of the pions.*

Energy of pions in GeV	Depth in g/cm²		
	168	672	1008
10	0.2	0.4	0.5
50	0.9	1.2	1.2
100	1.2	1.3	1.5
500	1.6	1.8	1.8
1000	2.2	2.6	2.8

of the actual exponents, because in the analysis used to deduce the values in Table II, regeneration of pions by pions was neglected. However, it can be concluded from Table II that for energies of pions > 50 GeV, $L_i < L_\pi < 3L_i$, from considerations of the foregoing formula for L_π.

To consider the effects of regeneration of pions by pions, we have taken $L_\pi = L_a = 120$ g/cm² as a possible upper limit to L_π. We have drawn in Fig. 2 a dashed curve, indicated by (*b*), which gives $\Omega(h)$ for this limit of L_π. $\Omega(h)$ is now obtained as outlined in Section **2** replacing L_i by L_π in the term representing loss of pions by nuclear interactions. We find that $\Omega(h)$ is increased to the extent of 8% at $h = 10$ and of 33% at $h = 0.1$. The production spectrum drawn in Fig. 3 gets reduced correspondingly. Thus the production spectrum is not much affected. It is worth pointing out that if $L_\pi = \infty$, $\Omega(h) = L_a/L_i = 1.71$. In this case the production spectrum runs parallel to the muon spectrum.

The intensity distribution of pions in the atmosphere gets appreciably affected depending on the value of L_π because of the function $\Lambda(y, h)$. For our choice of L_π the intensity of pions at sea level increases by about 40% at 10 GeV, while at 1000 GeV the intensity increases by a factor of 4.6. The above estimate at 10 GeV is, in all probability, an underestimate, because L_π could even be negative due to the flatness of the pion spectrum at this energy. Even if L_π is negative at this energy, the production spectrum is again not likely to be affected much because of relatively large decay probabilities at energies near and above 10 GeV. But the pion intensities at sea level could go up by a large factor so as to result in the values of the charged-to-neutral

ratio within brackets at 10 GeV in the second column of Table I approaching the values in the brackets at 50 GeV given in the next column (*).

It should be pointed out that the analysis carried out in this paper is subject to uncertainties in the absolute intensity and energy spectrum of muons at sea level and of nucleons at various depths in the atmosphere. These fluxes are not likely to be uncertain by more than a factor of two and since no claim is made to great accuracies in the absolute intensities of pions deduced here, because of the effect of regeneration of pions by pions, these uncertainties will not affect the main results obtained in this paper.

5. – Average inelasticity in collisions of high energy nucleons with air nuclei.

The relationship between the differential energy spectrum of pions at production and the differential primary nucleonic energy spectrum is of great significance. It is possible to interpret the relationship between these two spectra in terms of the average inelasticity and multiplicity in the collision of a nucleon with an air nucleus. In Fig. 3, the curve labelled N gives the differential energy spectrum of primary nucleons as has been used in Section 3. Comparison of the production spectrum of pions with the primary nucleonic spectrum shows that

$$N(E)\,\mathrm{d}E = kF_{\pi}(E)\,\mathrm{d}E\,,$$

where k is a numerical factor. If ε and n are the average inelasticity and multiplicity of energetic charged pions in the interaction of a nucleon of energy E', we can write

$$E = \varepsilon E'/1.5\,n$$

assuming equipartition of energy among the secondary pions; the factor 1.5 takes account of the production of neutral pions. If we further take that

$$N(E)\,\mathrm{d}E \propto \mathrm{d}E/E^{\gamma+1}$$

it follows that

$$k = n^{\gamma-1}\,1.5^{\gamma}/\varepsilon^{\gamma}\,.\,.$$

We have given in Table III the values of k at different energies read off from Fig. 3 and the calculated values of n for some values of ε, taking $\gamma = 1.5$.

(*) This estimate is made from the observation that the ratio $A(-10, 10)/A(6, 10) \approx 6$

TABLE III. – *Values of multiplicities of high energy pions produced in a nucleon-air nucleus collision to account for the ratio between the differential production spectrum of pions and the differential energy spectrum of nucleons at the top of the atmosphere.*

Energy in GeV	k	n		
		$\varepsilon = 0.7$	$\varepsilon = 0.5$	$\varepsilon = 0.3$
10	23	56	20	4.4
50	17	31	11	2.4
100	16	27	10	2.1
500	20	41	15	3.2
1000	34	115	42	9.1

We see from this Table that for $\varepsilon = 0.7$, the values of n are very large. Such large values of multiplicities will be inconsistent with the average multiplicities one finds in photographic emulsions ([7]) and cloud chambers ([10]). It should be remembered that n denotes roughly the multiplicity of high energy charged secondaries and this can be only about half the multiplicity one sees in high energy nuclear interactions. A reasonable value that can be expected for n at these energies is about ten ([7,10]). Thus we can say that ε lies between 0.3 and 0.5. The effect of regeneration of pions by pions changes little the production spectrum, as shown in Section **4**, and hence will not affect this estimate for ε.

GRIGOROV ([5]) has shown that, in the latitude sensitive energy region of the cosmic ray primaries, nucleons on the average do not multiply by collision with air nuclei. This will also be the case at higher energies, if transverse momenta of incident and target nucleons after collision are small ($\sim$ 1 GeV/c ([7])) and if ε is also small (*). We can then write

$$L_i/L_a = 1 - (1 - \varepsilon)^{\gamma} ,$$

which shows that our choice of $L_i = 70$ g/cm^2 and $L_a = 120$ g/cm^2 leads to a value of $\varepsilon \approx 0.4$, consistent with the estimate of ε given above. The value for L_i as high as 100 g/cm^2, suggested by BRENNER and WILLIAMS ([15]), is inconsistent with the analysis made in this Section.

6. – The zenith angular distribution of high energy muons underground.

The production spectrum of pions as drawn in Fig. 3, does not have the same value of the exponent as the exponent derived from the zenith angular

(*) We further assume that production of nucleon-antinucleon pairs is negligible and that nucleon-nucleon collisions are symmetrical in the C.M. system.

distribution of muons underground ([6]). While the former is about 2.3, the latter value is about 2.9 in the energy region (25 ÷ 50) GeV. This discrepancy can be understood if part of the pions originating in the atmosphere are produced by higher energy pions. If this is so, the analysis which leads to the exponent of the production spectrum from the angular distribution of muons underground has to be modified. The effect of regeneration of pions by pions on the vertical intensity distribution of pions in the atmosphere has been discussed in Section **4**. Here, as regards the analysis of zenith angular distribution of muons underground, the modification will involve rewriting the diffusion equation for the pions in the atmosphere with an explicit separation of the source function for pions into two parts: one part giving the source term for production by parent nucleons and the other part giving the source term for production by parent pions. This separation is impossible to make at the present time since it involves a knowledge of the nature of the interaction of pions with air nuclei. Instead, from the magnitude of the existing discrepancy, we shall estimate roughly the fraction of pions produced by the interactions of higher energy pions in a vertical column of the atmosphere, at a particular energy in the above mentioned energy range, assuming that the above explanation of the discrepancy is correct. We shall also discuss the implications of this estimate.

Let $\mu_v(E)\,\mathrm{d}\bar{E}$ be the differential intensity of muons in the vertical direction (determined by the rate of vertical absorption of muons), and let $\mu_\theta(\bar{E})\,\mathrm{d}\bar{E}$ be the differential intensity of muons in a direction inclined at an angle θ to the vertical (determined by the angular distribution of muon intensity). The excess rate of production of pions in the vertical column at an energy E, in interval $\mathrm{d}E$, as compared to a column inclined at an angle θ is given by

$$\frac{L_a}{L_i}\left\{\frac{\mu_v(\bar{E})\,\mathrm{d}\bar{E}}{\Omega(h)} - \frac{\mu_\theta(\bar{E})\,\mathrm{d}\bar{E}}{\Omega(h\sec\theta)}\right\},$$

provided that the distribution of pions at production is exponential in the atmosphere with a mean free path L_a. Regeneration of pions by pions will make the distribution of pions at production deviate from an exponential distribution. However, the estimate made here of the total rate of pion production in an entire column of the atmosphere to account for the observed intensity of muons at sea level will be insensitive to a slight departure from an exponential distribution of pions at production. Further, since we are interested only in estimating the relative proportion of pions generated by pions, to the total rate of production of pions in a vertical column in the atmosphere, the errors involved will be of no significance.

To evaluate the above expression, including the uncertainties involved, the integral intensities of muons underground, denoted by $I(h, \theta)$, where h is the

depth underground, is written as

$$I(h, \theta) = I(h, 0) \cos^n \theta ,$$

with

$$I(h, 0) \propto h^{-m} .$$

From this it follows that

$$\mu_\theta(\bar{E})\, \mathrm{d}\bar{E} = \frac{n}{m} \cos^{n-m} \theta \mu_v(\bar{E})\, \mathrm{d}\bar{E} ,$$

if the range-energy relation is assumed to be linear for the muons. This will be true for the energy range in which we are making the analysis. Choosing $\bar{E} = 44.7$ GeV, which value lies in the stated energy interval and suits convenience in the calculations, m was found to lie between 1.8 and 2.1 ([3]) and $n = 1.92 \pm 0.08$ ([6]). θ was chosen to be 60° which is the largest angle one could go to without introducing appreciable errors due to the spherical shell distribution of the atmosphere ([19]).

After correcting for μ-e decay losses in the atmosphere, the excess rate of production of pions at an energy of 59 GeV in the vertical column, as compared to a column inclined 60° to the vertical, turns out to be $(5.7 \div 8.5) \cdot 10^{-7} \cdot \cdot \mathrm{cm}^{-2} \cdot \mathrm{s}^{-1} \cdot \mathrm{sr}^{-1} \cdot \mathrm{GeV}^{-1}$. The total rate of pion production at energy E, in interval $\mathrm{d}E$, in a vertical column, is given by the expression

$$\frac{L_a}{L_i} \frac{\mu_v(\bar{E})\, \mathrm{d}\bar{E}}{\Omega(h)} .$$

This at 59 GeV is $(5.1 \pm 0.4) \cdot 10^{-6}$ cm^{-2} s^{-1} sr^{-1} GeV^{-1}. Thus about 10 to 20% of all pions produced in the vertical column at 59 GeV are produced by the excess of pion interactions in the vertical compared to 60°. In Fig. 5, we have plotted $(L_a/L_i) N(E)\, \mathrm{d}E$, which gives the total rate of interactions produced by all nuclear active particles in the atmosphere in a vertical column, and

$$\frac{\mu_v(\bar{E})\, \mathrm{d}\bar{E}}{\Omega(h)} \{L_a/L_i - \Omega(h \sec \theta)\} .$$

which gives the rate of pion interactions in a column inclined at θ to the vertical, for $\theta = 0°$ and 60° (*). We have also drawn there the difference

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([19]) D. JAKEMAN: *Can. Journ. Phys.*, **34**, 432 (1956).

(*) The curve for 60° is only approximate because the expression used to draw it is valid only if the rate of production of pions in the atmosphere remains independent of the inclination of the column considered. As found earlier, the rate of production of pions at 59 GeV falls by (10÷20)% at 60°. A correction of this order at 59 GeV and less at higher energies will not vitiate any of the conclusions arrived at in this Sect.

between the curves for 0° and 60°. We find from Fig. 5, that at least about 25% of all pions at 59 GeV produced in the vertical column should be produced by pions. This is because only about 40% of all pion interactions, at energies higher than 59 GeV but close to it, in the vertical column, are accounted for by the difference in the rate of pion interactions between 0° and 60°. Thus we come to the conclusion that regeneration of pions by pions is likely to be significant, which means that pions are not absorbed catastrophically in one collision with an air nucleus.

As regards the average energy of the parent pions which give rise to pions of energy 59 GeV, we can say that it lies anywhere higher than 59 GeV but not more than about twice this energy. The upper limit to the parent pion energy is set by demanding that only less than about one third of the pions of energy 59 GeV produced in a vertical column in the atmosphere at 59 GeV can arise from pion interactions. This limit is in turn set by demanding that L_π should be less than $3L_i$ (see Section **4**). From the analysis in Section **5**, we can expect a nucleon of energy ~ 1000 GeV to produce, on the average, pions of energy about 59 GeV. Hence we can conclude that pions need less energy to produce secondary pions of a given energy than nucleons. This is also evi-

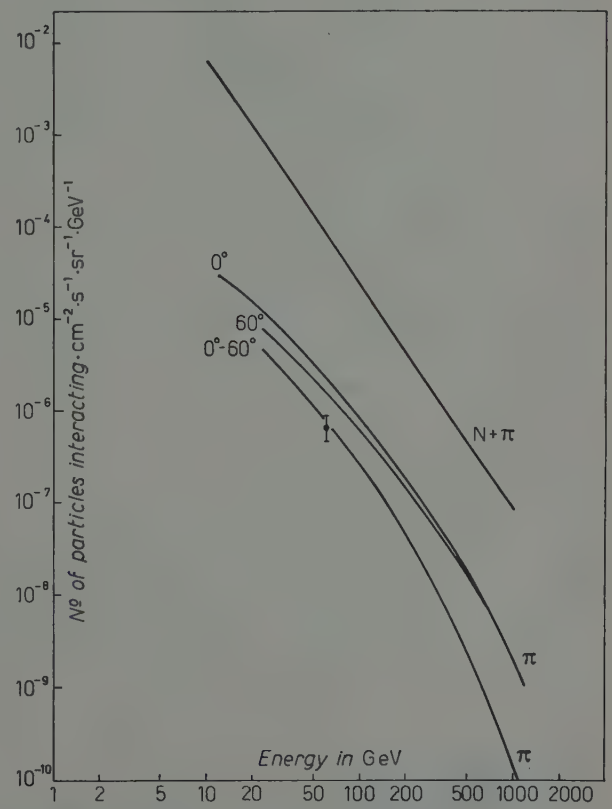

Fig. 5. – Rate of interactions of different particles in the atmosphere. $N+\pi$, total nuclear active particles in a vertical column; 0°, pions in a vertical column; 60°, pions in a column inclined 60° to the vertical; 0—60°, difference between a vertical column and a column inclined 60° to the vertical. The plotted point is the excess rate of production of pions at an energy of 59 GeV in a vertical column as compared to a column inclined 60° to the vertical, which is necessary to account for the observed differences in the vertical rate of absorption and the zenith angular dependence of muons underground.

dent from Fig. 5, where although the number of pion interactions at any energy is $\leqslant 5\%$ of the interaction rate of all nuclear active particles at the same energy in the vertical column, the pion interactions account for at least 25% of the pion production rate at 59 GeV. The point plotted in Fig. 5, when compared with the rate of pion interaction curve indicated by $(0 \div 60°)$ also reveals the high efficiency of regeneration of pions by pions. Our conclusion regarding the difference in the interaction characteristics of pions and nucleons in the production of pions agree with the findings of the Bristol group ([7]) that the « inelasticity » is close to unity for secondary « jets » while for primary « jets » the « inelasticity » is much less than unity. *Further, the analysis made above indicates that a high energy pion on colliding with an air nucleus produces on the average one or two secondary pions of energy comparable to the energy of the primary pion.*

Before concluding this Section, we would like to refer to a recent paper by CERVASI FIDECARO and others ([20]) wherein is reported an anomaly in the zenith angular dependence of nucleons at mountains altitudes. The anomaly consists in the zenith angular dependence of nucleons being not simply accounted for by a mass proportional absorption. This anomaly can be sought to be explained as due to the participation of unstable nuclear active particles in the development of the nuclear cascade in the atmosphere. Following the analysis of the zenith angular dependence of muons underground, as presented in this section, it looks reasonable to expect a zenith angle anomaly in the nucleonic component because of pion interactions in the atmosphere. But as shown in Section **5**, the inelasticity factor in nucleon-air nucleus collision is small and hence it will be difficult to account for the magnitude of the anomaly reported (*).

We have already indicated in Section **3** the importance of measuring accurately the ratio of charged to neutral particles among the nuclear active particles in the lower atmosphere as a function of energy to evaluate the regeneration characteristics of pions by pions. We wish to emphasize here that an accurate knowledge of the zenith angular distribution of muons underground and the zenith angular distribution of high energy nucleons at mountain altitudes will enable one to deduce the average characteristics of high energy

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([20]) M. CERVASI FIDECARO, G. FIDECARO, G. MARINI and L. MEZZETTI: *Nuovo Cimento*, **9**, 37 (1958).

(*) It has been shown by the authors of the paper quoted that the inelasticity coefficient of about 0.2 used in Budini and Molière's theory of nuclear cascade in the atmosphere ([21]) is unable to account for the zenith angle anomaly and that an inelasticity coefficient of 0.7 suggested by BRENNER and WILLIAMS ([15]) might explain the anomaly. But our analysis of muon intensity underground favours an inelasticity of about 0.4.

pion interactions in air (*). Interpretation of these data purely in terms of pion interaction characteristics in the air will be justified since direct observation of high energy nuclear interactions in photographic emulsions has revealed the non-copius production of particles other than pions ([7,8]). For the same reason, consideration of the origin of muons underground from decay of K-mesons ([4,6,23]) can be overlooked.

7. – Conclusions.

On the basis of the foregoing analysis, it is possible to conclude that the vertical intensity of pions must be a minor fraction of the vertical intensity of all nuclear active particles in the atmosphere in the energy range $(10^{10} \div 10^{12})$ eV. The fraction of energy lost in meson production by nucleons of high energies in the range $(10^{11} \div 10^{13})$ eV in collisions with air nuclei is between 0.3 to 0.5. These conclusions will be little affected by the uncertainties in the value of the interaction mean free path of high energy nucleons and pions in air and also by the uncertainties in the exact magnitude of the regeneration of pions by pions in the atmosphere. The estimate given above for the average inelasticity coefficient for collisions of high energy nucleons with air nuclei is consistent with an interaction mean free path in air of 70 g/cm² for the high energy nucleons assumed in the analysis. It is possible to interpret the discrepancies in the vertical rate of absorption and the zenith angular dependence of muons underground in terms of regeneration of pions by pions. This interpretation leads to the conclusion that pions need less energy than nucleons to produce secondary pions of a given energy in collisions with air nuclei.

* * *

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(*) It should be pointed out that the positive temperature effect of muons underground ([4,22]) is also a datum, the interpretation of which will be affected by the characteristics of the interaction of pions in the atmosphere.

([21]) W. Heisenberg: *Kosmische Strahlung* (Berlin, 1953), p. 367.

([22]) N. Sherman: *Phys. Rev.*, **93**, 208 (1954).

([23]) C. A. Randall and W. E. Hazen: *Nuovo Cimento*, **8**, 878 (1958).

Dr. R. R. DANIEL and Mr. D. S. NARAYAN who also read and discussed with us a preliminary draft of this paper. Mr. D. S. NARAYAN, in addition, suggested to us a simple procedure in the numerical evaluation of certain integrals involved in this paper and also pointed out to us some limitations in our treatment of the effects of regeneration of pions by pions. We also thank Dr. S. NARANAN with whom we had stimulating discussions and who drew the range-energy relation for muons underground, which we made use of.

RIASSUNTO (*)

Dallo spettro di energia dei muoni al livello del mare si calcola la distribuzione verticale dei pioni nell'atmosfera nel campo d'energia ($10^{10} \div 10^{12}$) eV. Si trova che a tutte le profondità nell'atmosfera i pioni rappresentano solo una frazione minore dell'intensità delle particelle nucleari attive nel suddetto campo d'energia. Il confronto dello spettro energetico dei pioni alla produzione collo spettro primario dei nucleoni porta al risultato che l'inelasticità media nelle collisioni dei nucleoni coi nuclei dell'aria, nel campo d'energia ($10^{11} \div 10^{13}$) eV sta fra 0.5 e 0.3 se si pone un limite ragionevole alle molteplicità dei mesoni prodotti in queste collisioni. Si fa rilevare che la generazione nell'atmosfera di pioni da parte di pioni può avere apprezzabile effetto sulla dipendenza dall'angolo zenitale dei muoni sottoterra e che, se così è, ciò porta al risultato che i pioni sono più efficaci nel produrre pioni di quanto non lo siano i nucleoni nelle collisioni coi nuclei dell'aria.

(*) *Traduzione a cura della Redazione.*

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Statistical Derivation of the Nuclear Rotational Energies.

K. Kumar

Tata Institute of Fundamental Research - Bombay (*)

(ricevuto il 21 Aprile 1959)

Summary. — The nucleus is treated as a Fermi gas under the constraint of a given angular momentum I. Its energy is expressed as a power series in I^2, and the rotational and rotation-vibration interaction terms are identified. The interparticle interactions do not influence the rotational energies whereas they have important influence on the rotation-vibration energies. Several models for angular momentum production are considered. The experimental trend of the moments of inertia is reproduced by a model in which only nucleons outside a certain « core » produce the total angular momentum. The vibrational frequency is insensitive to such models. The surface effects and the influence of velocity dependent forces are also taken into account.

1. – Introduction.

The problem of the rotational states in nuclei has attracted considerable attention in recent years ([1-4]). The problem has two aspects. The first, connected with the spins, parities and the ratios of the energies of the excited states seems to be fairly well understood. Even models starting from very different ideas are in agreement as far as these properties are concerned. The second aspect is connected with the absolute magnitudes of the energies and with the vibrational perturbation of the main rotational terms. The main

(*) Part of this work was done while the author was at the Department of Physics, Purdue University, Lafayette, Indiana.

([1]) S. A. Moszkowski: *Encycl. Phys.*, **39**, 411 (1957).

([2]) D. R. Inglis: *Nucl. Phys.*, **8**, 125 (1958).

([3]) L. Meichsner: *Nucl. Phys.*, **8**, 493 (1958).

([4]) H. J. Lipkin: *Nucl. Phys.*, **8**, 421 (1958).

problem here is to account for the empirical values of the moments of inertia. The idea common to all previous approaches ([1-4]) is to find a connection between a shell-model type of description and the collective model. Depending on the approximation and point of view a variety of answers may be obtained. The next step is to study the effect of internucleonic interactions on the moment of inertia. One treats the interaction in a shell-model scheme and then goes over to the collective description.

In the present work an alternative approach to the second aspect of the problem is advanced. It was shown by SESSLER and FOLEY ([5]), is their statistical treatment of an atom, that the requirement that an atom (Fermi gas) possess a net angular momentum leads to a rotation of the atom as a whole. This method applied to nuclei gives a simple way of introducing collective rotations. The question of the connection between shell and collective models does not arise here. The internucleonic interactions are then taken into account and their effect on the rotational and rotation-vibration energies is discussed.

In the next section the basic method for calculating the rotational and vibrational-rotational energies on the statistical model is given. It is shown that the rotational energy ($\sim I(I+1)$) is independent of the strength of two-particle interactions while the vibration-rotation energy ($\sim I^2(I+1)^2$) is considerably influenced by them. The moment of inertia is found to be sensitive only to the way in which the angular momentum I is produced by the nucleons, whereas the vibrational frequency is shown to be insensitive to this. In Section **3** various models are discussed for the moment of inertia with the view of reproducing the observed trends. In Section **4** the effects of the finite size of the nucleus and velocity dependence of nuclear forces are taken into account. It is shown that the velocity dependence does not give as large a factor to the moment of inertia as one obtains from simple substitution of the effective for the actual nucleon mass in the final expressions. When all such effects are taken into account and an appropriate model for the mechanism of angular momentum production is introduced, agreement with experimental trends is obtained. One does not expect a statistical model to allow a determination of spins and parities of the states. The present investigation takes these things for granted but provides a new way of looking at the energies.

2. – Rotational energy and vibration-rotation interaction.

Following SESSLER and FOLEY ([5]) we observe that the center of the Fermi sphere in the momentum space has to be displaced from the origin by an amount $\boldsymbol{D}_n$ in order to produce a net angular momentum $\boldsymbol{I}_n$ of a Fermi gas

([5]) A. M. SESSLER and H. M. FOLEY: *Phys. Rev.*, **96**, 366 (1954).

at zero temperature, say of neutrons. It is given by the relation

$$\boldsymbol{I}_{\mathrm{n}} = \int (\boldsymbol{r} \times \boldsymbol{D}_{\mathrm{N}}(\boldsymbol{r})) \, 2N(k) \, \mathrm{d}^3k \, \mathrm{d}^3r \,, \tag{1a}$$

$$= \int (\boldsymbol{r} \times \boldsymbol{D}_{\mathrm{n}}) \, \varrho_{\mathrm{n}}(\boldsymbol{r}) \,, \tag{1b}$$

where $N(k)$ is the density of momentum states and $\varrho_{\mathrm{n}}(\boldsymbol{r})$ the neutron density (*).

The protons in the nucleus are treated as another independent Fermi gas. A similar expression for its angular momentum, $\boldsymbol{I}_{\mathrm{p}}$ can also be written. The total angular momentum of the nucleus is now written as

$$\boldsymbol{I} = \boldsymbol{I}_{\mathrm{n}} + \boldsymbol{I}_{\mathrm{p}} \,. \tag{2}$$

Eqs. (1) and (2) are based on the assumption that a simple addition of the angular momenta arising from different parts of the nucleus takes place, The significance of $\boldsymbol{D}$ must be noted. SESSLER and FOLEY [5] have pointed out that if one follows the usual procedure of maximizing the probability of distribution with the constraint of constant angular momentum together with the usual constraints of total energy and number of particles, one gets a momentum distribution which is equivalent to a Fermi momentum sphere displaced from the origin. The co-ordinates of the centre of the momentum sphere are given by the Lagrange parameters associated with the constraint of angular momentum. These are denoted by $\boldsymbol{D}_{\mathrm{n}}$. Then the equ. (1) and (2) are written down following SESSLER and FOLEY. $\boldsymbol{D}$'s are so far quite arbitrary. From a different point of view also one may expect that starting from a knowledge of the interactions it will be possible to express the total angular momentum of a nucleus (or an atom) as an integral over the density as done in (1). Then $\boldsymbol{D}$'s will no longer be arbitrary but will be derived from a knowledge of interactions. This is of course not possible at present. One therefore prescribes some way of determining $\boldsymbol{D}$'s. SESSLER and FOLEY [5] have used a variational principle (v.p.) in which the energy is minimized to determine $\boldsymbol{D}$. This is a reasonable procedure and we shall adopt it in the present work. It is quite clear, however, that what we achieve by equ. (1) and (2) and the v.p. for $\boldsymbol{D}$ is only a certain model for angular momentum coupling. It is possible to construct other models by supplementing the v.p. for $\boldsymbol{D}$ with certain other restrictions on its form.

(*) Wherever the variables of integration are omitted they are to be understood as the three-dimensional volume element d^3r or d^3r'.

To illustrate the general features of the method we use in this section the model of angular momentum coupling in which $\boldsymbol{D}_n$ and $\boldsymbol{D}_p$ are to be determined entirely by the variational procedure. No other restriction on the form of $\boldsymbol{D}_n$, $\boldsymbol{D}_p$ is placed. We shall calculate the energies up to the term $\sim I^4$, the so-called vibration-rotation terms.

We shall put $1 = \hbar = e = m = 1837\ m_e \cong m_n \cong m_p$. In these units

$$\text{unit length} = 2.88 \cdot 10^{-12}\ \text{cm}$$

$$\text{unit energy} = 4.998 \cdot 10^{-2}\ \text{MeV} \quad \text{or} \quad 1\ \text{MeV} = 20\ \text{n.u.}$$

The name nuclear units and abbreviation n.u. is used for these units.

The total energy $\mathcal{E}$ of the nucleus will be the sum of the following kinetic and potential energies:

$$T_n = \int (c\varrho_n^{\frac{5}{3}} + \tfrac{1}{2} D_n^2 \varrho_n)\,, \tag{3a}$$

$$T_p = \int (c\varrho_p^{\frac{5}{3}} + \tfrac{1}{2} D_p^2 \varrho_p)\,, \tag{3b}$$

$$\overline{V}_{pp} = \tfrac{1}{2} \int V_{pp}(\boldsymbol{r}', \boldsymbol{r})\, \varrho_p(\boldsymbol{r})\, \varrho_p(\boldsymbol{r}')\,, \tag{3c}$$

$$\overline{V}_{np} = \int V_{np}(\boldsymbol{r}', \boldsymbol{r}')\, \varrho_n(\boldsymbol{r})\, \varrho_p(\boldsymbol{r}')\,, \tag{3d}$$

$$\overline{V}_{nn} = \tfrac{1}{2} \int V_{nn}(\boldsymbol{r}', \boldsymbol{r})\, \varrho_n(\boldsymbol{r})\, \varrho_n(\boldsymbol{r}')\,, \tag{3e}$$

where $c = 3(3\pi^2)^{\frac{2}{3}}/10$; the subscripts p and n refer to protons and neutrons respectively. Other notations are standard.

The unknown functions D_n, D_p, ϱ_n, ϱ_p of equation (3) are now found by minimizing the total energy $\mathcal{E}$, under the constraints of constant numbers of particles and constant value of $(I)^2$ and I_z. In the usual method of Lagrange multipliers the quantity to be varied is

$$\mathcal{L} = \mathcal{E} - a_p \int \varrho_p - a_n \int \varrho_n - \frac{\lambda}{2} \boldsymbol{I} \cdot \boldsymbol{I} - \lambda' \boldsymbol{e}_z \cdot \boldsymbol{I}\,, \tag{4}$$

where a_p, a_n, λ and λ' are constants to be determined later from the equations representing the constraints, $\boldsymbol{e}_z$ is the unit vector in the z-direction.

Variations are taken with reference to the eight unknown functions and the requirement

$$\delta \mathcal{L} = 0\,, \tag{5}$$

with reference to each of them determines their respective forms. The re-

sults are

$$\boldsymbol{D} = \boldsymbol{D}_n = \boldsymbol{D}_p = -\boldsymbol{r}\times(\lambda\boldsymbol{I} + \lambda'\boldsymbol{e}_z)\,, \tag{6}$$

$$\frac{5c}{3}\varrho_n^{\frac{5}{3}}(\boldsymbol{r}) = \tfrac{1}{2}D_n^2 + a_n - \int[V_{np}\varrho_p(\boldsymbol{r}') + V_{nn}\varrho_n(\boldsymbol{r}')]\,, \tag{7}$$

$$\frac{5c}{3}\varrho_p^{\frac{5}{3}}(\boldsymbol{r}) = \tfrac{1}{2}D_p^2 + a_p - \int[V_{np}\varrho_n(\boldsymbol{r}') + V_{pp}\varrho_p(\boldsymbol{r}')]\,, \tag{8}$$

From (1), (2) and (6) λ, λ', and D^2 are determined. If the densities ϱ_n and ϱ_p are cylindrically symmetric and if the vector model is assumend for I, *i.e.*, if $I_x = I_y = \frac{1}{2}$ we have

$$\lambda^{-1} = \int r^2\varphi(\theta)\varrho_A(\boldsymbol{r})\,, \tag{9}$$

$$\lambda' = I_z\left(\lambda - \int(r^2\varphi(\theta) - z^2)\varrho_A\right)^{-1}, \tag{10}$$

$$D^2 = D_n^2 = D_p^2 = \lambda^2 I^2 r^2\varphi(\theta)\,, \tag{11}$$

where

$$\varphi(\theta) \equiv \varphi_r(\theta) = \tfrac{1}{2}(1 + \cos^2\theta) \tag{12}$$

and

$$\varrho_A(\boldsymbol{r}) = \varrho_n(\boldsymbol{r}) + \varrho_p(\boldsymbol{r})\,.$$

We shall confine ourselves to even-even nuclei only. In that case the ground state has $I = I_z = 0$ and for excited states $I_z = 0$. Thus $\lambda' = 0$ in all the cases to be treated here.

Equation (9) shows that λ^{-1} has the form of the moment of inertia of the nucleus. With $\varphi(\theta)$ given by (12) it corresponds to a rigid moment of inertia. $\varphi(\theta)$ will be referred to as the flow-function.

Since the equations (7) and (8) are of the same form we shall make the reasonable assumption that $\varrho_n(\boldsymbol{r})$ and $\varrho_p(\boldsymbol{r})$ are proportional to each other and that

$$-\int(V_{np}\varrho_n(\boldsymbol{r}') + V_{pp}\varrho_p(\boldsymbol{r}')) = \int V_p(\boldsymbol{r}, \boldsymbol{r}')\varrho_p(\boldsymbol{r}')\,, \tag{13a}$$

$$-\int(V_{np}\varrho_p(\boldsymbol{r}') + V_n\varrho_n(\boldsymbol{r}')) = \int V_n(\boldsymbol{r}, \boldsymbol{r}')\varrho_n(\boldsymbol{r}')\,. \tag{13b}$$

The equations (7) and (8) are decoupled by this assumption and

$$\frac{5c}{3}\varrho_n^{\frac{5}{3}}(\boldsymbol{r}) = a_n + \frac{D_n^2}{2} + \int V_n(\boldsymbol{r}, \boldsymbol{r}')\varrho_n(\boldsymbol{r}')\,. \tag{14}$$

For the ground state of even-even nuclei $D^2 = 0$ from equation (11). Then the change in density $\delta\varrho_n(\boldsymbol{r})$ in going from the ground state to an excited state with total angular momentum I is given by:

$$\frac{10c}{9}\,\varrho_{0n}^{-\frac{1}{3}}(\boldsymbol{r})\delta\varrho_n(\boldsymbol{r}) = \frac{1}{2}D_n^2 + \delta a_n + \int V_n(\boldsymbol{r}, \boldsymbol{r}')\,\delta\varrho_n(\boldsymbol{r}')\,, \tag{15}$$

where $\varrho_{0n}(\boldsymbol{r})$ is the ground state density and δa_n, the change in the constant a_n, is determined from the condition

$$\int \delta\varrho_n(\boldsymbol{r}) = 0\,, \tag{16}$$

which is a consequence of the constraint of constant number of particles

$$\int \varrho_n(\boldsymbol{r}) = N\,.$$

The neutron contribution to the energy of the excited state may now be calculated from equation (3), using (7), (8), (11), (13) and (15). In terms of ϱ_{0n} and ϱ_{0p} and D of the excited state we have, to second order

$$E_{I(n)} = \frac{1}{2}\int D^2\varrho_{0n} + \frac{3}{4}\int D^2\,\delta\varrho_n\,. \tag{17}^*$$

Adding the neutron and proton contributions, we have, because of the linear dependence on $\delta\varrho$ and ϱ, the total energy

$$E_I (= E_{I(n)} + E_{I(p)}) = \frac{1}{2}\int D^2\varrho_{0A} + \frac{3}{4}\int D^2\,\delta\varrho_A\,, \tag{18}$$

where $\delta\varrho_A = \delta\varrho_n + \delta\varrho_p$. Note that from (9) and (11)

$$D^2 = \lambda_0^2 I^2 r^2\varphi(\theta)\left[1 - 2\lambda_0\int r^2\varphi(\theta)\,\delta\varrho_A + \ldots\right], \tag{19}$$

where $\lambda_0^{-1} = \int r^2\varphi(\theta)\varrho_{0A}(\boldsymbol{r})$, the « moment of Inertia » of the nucleus in the ground state.

Since $\delta\varrho_A \sim \lambda^2 I^2$ we have up to terms in I^4:

$$E_I = \frac{1}{2}\lambda_0^2 I^2\int r^2\varphi(\theta)\varrho_{0A}(\boldsymbol{r}) - \frac{1}{4}\lambda_0^2 I^2\int r^2\varphi(\theta)\,\delta\varrho_A(\boldsymbol{r})\,. \tag{20}$$

(*) It is easy to show that in this method a one body potential energy of the form $\overline{V} = \int \mathcal{V}_n(\boldsymbol{r})\varrho_n(\boldsymbol{r})$ will not contribute to the energy of the excited state.

Replacing I^2 by its quantum mechanical value $I(I+1)$ we get the rotational energy from the first term of (20)

$$(21) \qquad (E_I)_{\text{rot}} = \frac{I(I+1)}{2\int r^2 \varphi(\theta)\, \varrho_{0A}(\boldsymbol{r})} .$$

It is seen that the rotational energy is determined entirely by the *ground state density* and the flow-function $\varphi(\theta)$. This $\varphi(\theta)$ is determined by the angular momentum coupling (Eq. (11)). Further Eq. (21) does not contain $\delta\varrho_A$ in it and is therefore independent of the interaction $V(\boldsymbol{r}, \boldsymbol{r}')$. It is important to note that here the rotational energy is determined by the angular momentum coupling scheme and is otherwise independent of the two particle interaction strength. No such clear cut separation of these two causes occurs in other treatments of this problem and comparisons may not be justified. However, it seems to point towards a scheme in which the angular momentum is produced by the combined effect of only a relatively small number of nucleons. The present model, in which $\boldsymbol{D}$ is determined entirely by the variational principle, gives rise to rigid rotation and is thus unacceptable. Before introducing models that give better agreement with experimental data we analyse the behaviour of the second order terms.

The second order terms in equation (20) represent the vibration-rotation interaction energy:

$$(22) \qquad (E_I)_{\text{rot-vib}} = -\frac{1}{4}\lambda_0^2 I^2(I+1)^2 \int r^2 \varphi(\theta)(\delta\varrho_{\text{n}} + \delta\varrho_{\text{p}}) ,$$

$\delta\varrho_{\text{n}}$ is determined from equation (15) and (16) and there are corresponding equations for $\delta\varrho_{\text{p}}$. To illustrate the behaviour of this term we solve the equation (15) for the case where

$$(23) \qquad V_{\text{n}}(\boldsymbol{r}, \boldsymbol{r}') = V'_{0\text{n}}\,\delta(\boldsymbol{r} - \boldsymbol{r}') ,$$

where $V'_{0\text{n}}$ is a constant of the dimensions energy density. The solution of (15) and (16) up to terms in $\lambda_0^2 I^2$ is

$$(24) \qquad \delta\varrho_{\text{n}}(\boldsymbol{r}) = -\frac{9}{20c}\lambda_0^2 I^2 \tilde{\varrho}_{0\text{n}}^{\frac{1}{3}} \left[\frac{\int r^2 \varphi \varrho_{0\text{n}}^{\frac{1}{3}}}{\int \varrho_{0\text{n}}^{\frac{1}{3}}} - r^2\varphi(\theta)\right] ,$$

where

$$(25) \qquad \tilde{\varrho}_{0\text{n}}^{\frac{1}{3}}(\boldsymbol{r}) = \varrho_{0\text{n}}^{\frac{1}{3}}(\boldsymbol{r}) \left[1 - \frac{9}{10c} V'_{0\text{n}} \varrho_{0\text{n}}^{\frac{1}{3}}\right]^{-1} .$$

Substituting in (22) we have

$$(26) \qquad (E_I)_{\text{rot-vib}} = -\frac{9}{80c}\lambda_0^4 I^2(I+1)^2 \left[\int r^4 \varphi^2 \tilde{\varrho}_{0A}^{\frac{1}{3}} - \frac{(\int r^2 \varphi \tilde{\varrho}_{0A}^{\frac{1}{3}})^2}{\int \tilde{\varrho}_{0A}^{\frac{1}{3}}}\right] 2^{\frac{2}{3}} ,$$

where we have assumed that $\varrho_{0A} = 2\varrho_{0n} = 2\varrho_{0p}$. Actually, the non-linear dependence of $\delta\varrho_{0A}$ on ϱ_{0n} and ϱ_{0p} introduces a dependence on neutron excess but we ignore those terms.

From (25) and (26) we see that the vibration-rotation energy depends on the residual two particle interaction. Since V'_{0n} has been defined to be positive the effect is to increase the vibration-rotation contribution to the energy. To further illustrate this point we adopt a constant density model for the nucleus. The conclusions are found not to change much when more realistic, smooth density distributions are taken. In this case the analogue of (25) which will be used in (26) is given by

$$\tilde{\varrho}_{0A}^{\frac{1}{3}} = \varrho_{0A}^{\frac{1}{3}}[1 - 0.03V_0]^{-1}, \tag{27}$$

where V_0 is an average potential energy of residual two particle interactions in MeV.

For constant density inside a spheroid of boundary given by

$$r = R_0 f \equiv R_0(1 + \alpha P_2(\cos\theta)), \tag{28}$$

(26) reduces to

$$(E_I)_{\text{rot-vib}} = -\frac{9}{80c}\lambda_0^4 I^2(I+1)^2 2^{\frac{2}{3}}\tilde{\varrho}_A^{\frac{1}{3}}\frac{R_0^7}{7}2\pi\left[\langle\varphi^2 f^7\rangle - \frac{21}{25}\frac{\langle\varphi f^5\rangle^2}{\langle f^3\rangle}\right], \tag{29}$$

where

$$\langle\varphi^n f^n\rangle = \int_{-1}^{+1}\varphi^n(\theta) f^m(\theta)\,\mathrm{d}(\cos\theta). \tag{30}$$

Comparing with the expression in terms of the frequency of vibration ω [6],

$$(E_I)_{\text{rot vib}} = -2\left(\frac{1}{\hbar\omega}\right)^2\left(\frac{\hbar}{\Im}\right)^3 I^2(I+1). \tag{31}$$

One obtains for the vibrational frequency ω, which is an average of β and γ-vibrations

$$\omega^2 = \frac{160c}{9}\cdot\frac{7}{5}\left(\frac{\varrho_{A0}}{2}\right)^{\frac{2}{3}} R_0^{-2}(1 - 0.03V_0)\frac{\langle\varphi f^5\rangle}{\left[\langle\varphi^2 f^7\rangle - \frac{21}{25}\frac{\langle\varphi f^5\rangle^2}{\langle f^3\rangle}\right]}. \tag{32}$$

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[6] A. Bohr and B.R. Mottelson: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **26**, no. 16 (1953); A. Bohr: *Rotational States of Atomic Nuclei* (Copenhagen, 1954).

Putting in numerical values for ϱ_{0A} and $R_0 = 1.2A^{\frac{1}{3}} \cdot 10^{-13}$ cm,

$$\omega \cong 350 A^{-\frac{1}{3}}(1 - 0.03 V_0)^{\frac{1}{2}} . \tag{33}$$

The dependence of ω on the deformation parameter is very weak due to the fact that it appears only in angular integrals which in turn occur only as a ratio in the expression for ω, Eq. (32). Without interactions, *i.e.* with $V_0 = 0$, a value $\omega \cong 62$ MeV is obtained. It is about 10 to 60 times larger than the empirical values obtained by fitting the experimental rotational energies to a form ([1,6]):

$$E_I = \frac{I(I+1)}{2\Im} - \frac{2I^2(I+1)^2}{\omega^2 \Im^3} . \tag{34}$$

It is quite possible to try to explain the coefficients in the two terms independently but that procedure will not take into account the physical connection between the two terms.

From (33) it is seen that in order to bring ω to its observed value the interaction potential $V_0 \cong 30$ MeV it needed. This is a reasonable value. It has thus been shown that interactions are very important for vibration-rotation interactions. Their effect is to make the nucleus « soft » for vibrations. On the other hand the frequency of vibration, ω, is almost independent of deformation and the angular momentum coupling scheme (*i.e.* of the flow function $\varphi(\theta)$). We base this remark on explicit calculations for several models to be reported below with a uniform density distribution. The reason for insensitivity to the density distribution is partly the following. The density distribution for a spheroidal nucleus is obtained from that of a spherical nucleus by replacing r/R_0 in the latter by r/R, where $R = R_0 f(\theta)$. Any integral involving the function $\tilde{\varrho}^{\frac{1}{3}}$ of $\varrho(r)$ and $\varphi(\theta)$ can be expressed as follows:

$$\iiint \tilde{\varrho}^{\frac{1}{3}}(\boldsymbol{r}) \varphi^a(\theta) r^{za} \, \mathrm{d}^3 r = 2\pi R_0^{2a+3} \left[\int \tilde{\varrho}^{\frac{1}{3}}(x) x^{2a+2} \, \mathrm{d}x \right] \langle \varphi^a f^{2a+3} \rangle , \tag{35}$$

where x is a dimensionless variable with the range $\infty \geqslant x \geqslant 0$. Unless there is something capricious about $\tilde{\varrho}^{\frac{1}{3}}(x)$ the equ. (29) will continue to give a good approximation to (26).

It may be recalled that in the usual theory ([1,6,7])

$$\omega = \left(\frac{C_2}{B_2} \right)^{\frac{1}{2}} , \tag{36}$$

([7]) G. M. Temmer and N. P. Heydenberg: *Phys. Rev.*, **104**, 967 (1956).

where C_2 and B_2 are constants occuring in the collective Hamiltonian

$$H_{\text{coll}} = \tfrac{1}{2}\sum_{\mu} B_2 |\dot{\alpha}_\mu|^2 + \tfrac{1}{2} C_2 \sum_{\mu} |\alpha_\mu|^2 , \tag{37}$$

C_2 is the so-called surface tension and B_2 has the significance of a mass. These two terms have been found to be quite sensitive to the shell structure ([1,7]). Naturally one cannot explain this shell dependent behavior from a statistical treatment. However, the statistical method may apply to ω in virtue of Equ. (36) and the fact that both C_2 and B_2 seem to have similar dependence on the shell structure.

3. – Models for the moment of inertia.

It has been shown above that the moment of inertia, which is given by

$$\Im = \lambda_0^{-1} = \int r^2 \varphi(\theta) \varrho_{0A}(\boldsymbol{r}) , \tag{38}$$

depends only on the angular momentum coupling. The model considered above gives rigid rotation. However, empirically the moments of inertia are more sensitive to deformation and are significantly less than the rigid values. Because of this it has long been suspected ([8]) that the rotational states arise due to the co-operation of a relatively small number of particles. This can be given a concrete expression in the present model by writing, instead of equ. (2)

$$\boldsymbol{I} = \int (\boldsymbol{r} \times \boldsymbol{D}) \eta_i(\boldsymbol{r}) \varrho_A(\boldsymbol{r}) , \tag{40}$$

where $\eta_i(\boldsymbol{r})$ is a function which picks out the parts of $\varrho_A(\boldsymbol{r})$ that contribute to the angular moment. A full specification of $\eta_i(\boldsymbol{r})$ and the prescription for determining $\boldsymbol{D}$ constitutes a model. The subscript of η_i distinguishes η's of different models. Following models may be mentioned

a) $\boldsymbol{D}$ to be determined by v.p. and $\eta_r = 1$ has been treated above. It gives rigid rotation.

b) $\boldsymbol{D}$ to be determined by v.p. and

$$\begin{cases} \eta_b = 0 & r < R_0(1 - \varepsilon_b) , \\ \eta_b = 1 & r > R_0(1 - \varepsilon_b) , \end{cases} \tag{41}$$

([8]) K. W. Ford and D. L. Hill: *Ann. Rev. Nuc. Sc.*, **5**, 25 (1955).

where $\varepsilon_b \leqslant \alpha/2$, corresponds to rigid rotation of matter outside a spherical inert core (prolate shapes only).

c) $\boldsymbol{D}$ to be determined by v.p. and

$$(42) \qquad \begin{cases} \eta_c = 0 & r < R_0 \varepsilon_c f \,, \\ \eta_c = 1 & r > R_0 \varepsilon_c f \,, \end{cases}$$

where $\varepsilon_c < 1$, corresponds to rigid rotation of matter outside a spheroidal inert core of same shape as the nucleus.

d) $\boldsymbol{D} = D\boldsymbol{e}_z$, D to be determined by v.p. and $\eta_d = 1$ corresponds to a quasi-laminar flow with all the nucleus participating.

e) $\boldsymbol{D} = D\boldsymbol{e}_z$, D to be determined by v.p. and

$$(43) \qquad \begin{cases} \eta_e = 0 & r < R_0(1 - 0.5\alpha - \varepsilon_e \alpha^2) \,, \\ \eta_e = 1 & r > R_0(1 - 0.5\alpha - \varepsilon_e \alpha^2) \,, \end{cases}$$

which corresponds to a flow of the same type as in *d*) but only the matter outside the core defined by (43) takes part in the flow.

For all these models using (40) in Equ. (4) and the v.p.

$$(44) \qquad \boldsymbol{D} = -\boldsymbol{r} \times (\lambda \boldsymbol{I} + \lambda' \boldsymbol{e}_z) \eta_i \,,$$

$$(45) \qquad D^2 = \lambda^2 I^2 r^2 \varphi_i(\theta) \eta_i^2 \,.$$

For the models *a*), *b*) and *c*)

$$(46) \qquad \varphi_i(\theta) = \varphi_r(\theta) = \tfrac{1}{2}(1 + \cos^2 \theta)$$

and for the models *d*) and *e*)

$$(47) \qquad \varphi_i(\theta) = \varphi_l(\theta) = \sin^2 \theta \,.$$

The moment of inertia for the models is then written schematically

$$(48) \qquad \Im_i = \Im_r \frac{\langle \varphi_i(\theta) \eta_i^2 f^5 \rangle}{\langle \varphi_r(\theta) f^5 \rangle} \,,$$

$\Im_a = \Im_r$ and hence model *a*) is ruled out.

$\Im_b$ cannot be made to fit emperical data whatever form one may choose for ε_b as function of α or otherwise.

$\Im_c$ can be made to fit the data somewhat by putting $\varepsilon_c = (1 - 0.5\alpha) \cdot 0.84$ (See Fig. 1 curve (*i*) and also Equ. (54)). This is one of the many things one may try and is not satisfactory.

$\Im_d \cong \Im_r(1 - 1.5\,\alpha)$ and is ruled out.

$\Im_e$ gives the best fit to the data if $\varepsilon_e = 2$. See fig. 1 curve (ii).

Models *d*) and *e*) give rise to a peculiar flow, equ. (47). Perhaps one should not take it literally. But, the true physical picture, if it is at all possible, may be expected to be somewhat closer to this rather than to the rigid flow of Equ. (48) in view of the success of this model. The conclusion one can be more sure of is the necessity of an inert core as far as the angular momentum is concerned.

In contrast to the important difference that these models make to the rotational energy they give practically indistiguishable results for vibration-rotation energy. The finer details of deformation dependence of (29) are indeed different but they are also unimportant.

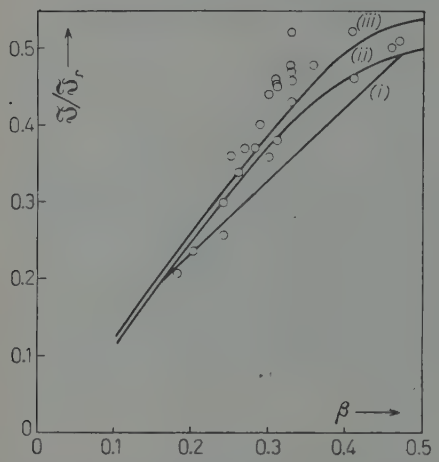

Fig. 1. – The moment of inertia on statistical model. Curve (i): model (*c*), with effective mass and surface corrections, $\varepsilon_c = (1 - 0.5\alpha)$. Curve (ii): model (*e*), without effective mass and surface corrections, $\varepsilon_e = 2$. Curve (iii): model (*e*) with surface and effective mass corrections, $\varepsilon_e = 3$. The circles are obtained using the experimental data given in ref. ([11]) $\alpha = 0.63\beta(1 - 0.16\beta + \ldots)$.

4. – Surface and effective mass effects.

Surface effects may be partly taken care of by using for the density of momentum states ([9])

$$N(k)\,\mathrm{d}^3k = \frac{\mathrm{d}^3k}{(2\pi)^3}\left(1 - \frac{k_0}{k_F}\right), \tag{49}$$

([9]) D. L. Hill and J. A. Wheeler: *Phys. Rev.*, **89**, 1102, (1953); K. A. Brueckner: *Phys. Rev.*, **96**, 508 (1954).

where $k_0 = 3\pi/4R_0$ for the case of the nucleus. The effect on the kinetic energy is to multiply the expressions (3*a*) and (3*b*) by a factor

$$\xi_S^{-1} = (1 + k_0/4k_F) \tag{50}$$

and the expressions of the angular momentum (16) and (40) are multiplied by a factor

$$\eta' = (1 + k_0/2k_F) ; \tag{51}$$

in all other expressions η' occurs in exactly the same way as $\eta_i(\boldsymbol{r})$ does. The Fermi momentum k_F may be assumed constant inside the nucleus, hence

$$k_0/k_F = 3\pi/4R_0(3\pi^2\varrho_n)^{\frac{1}{3}} \cong 1.23\,N^{-\frac{1}{3}} . \tag{52}$$

Effective mass correction can be taken into account by taking another factor ξ_m^{-1} with the kinetic energy expressions (3*a*) and (3*b*). To be correct this should be a function of $\boldsymbol{r}$ and occur inside the integral sign ([10]). The proce dure for calculation of energies goes through without much change. Finally,

$$\Im_i = \int r^2 \varphi_i(\theta)\, \eta_i^2(\boldsymbol{r})\, \eta'^2 \xi_S\, \xi_m\, \varrho_{0A}(\boldsymbol{r}) . \tag{53}$$

This may be expressed to a good approximation by

$$\Im_i = \bar{\xi} \int r^2 \varphi_i(\theta) \eta_i^2(\boldsymbol{r})\, \varrho_{0A}(\boldsymbol{r}) , \tag{54}$$

$\bar{\xi}$ is a constant determined by taking appropriate averages. For ξ_m the potential model of Frahn and Lemmer was used and a value of $\bar{\xi} = 0.84$ was found. $\bar{\xi}$ is not a free parameter but is determined from the density of the nuclear matter Eq. (52), and the analysis of the many body nuclear problem ([10]). When this factor is used then in model *e*) with $\varepsilon_e = 3$ gives the curve (iii) in Fig. 1. This we consider to be the best curve. The same factor has also been used with other models. These corrections do not affect the vibration-rotation energy very much.

([10]) H. A. Bethe: *Phys. Rev.*, **103**, 1353 (1956); W. E. Frahn and R. H. Lemmer: *Nuovo Cimento*, **5**, 1565 (1957); **6**, 1221 (1957).

([11]) K. Alder and others: *Rev. Mod. Phys.*, **28**, 432 (1956), Table V2.

5. – Conclusions.

A statistical model is developed which gives a consistent account of the rotational spectra of nuclei including the vibration-rotation interaction effects. It is shown that in order to obtain correct moments of inertia it is necessary to assume the presence of an « inert core » in the nucleus which does not participate in producing the angular momentum of the rotational states. The short range interactions among the nuclei have no effect on their moments of inertia. Further it seems that the flow in the active outer regions must deviate significantly from a rigid flow. On the other hand vibrational frequency is practically independent of the flow pattern and deformation. It is, however, strongly dependent on the interaction strength. In the δ-function limit a strength $V_0 = 30$ MeV gives correct results.

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It is a pleasure to acknowledge the very helpful discussions with Dr. D. C. Peaslee on various aspects of this work.

RIASSUNTO (*)

Si tratta il nucleo come un gas di Fermi con l'imposizione di un dato impulso angolare I. Si esprime la sua energia come una serie di potenze in I^2 e si identificano i termini d'interazione rotazionali e vibrorotazionali. Le interazioni fra particelle non influiscono sulle energie rotazionali, mentre agiscono in modo rilevante sulle energie vibrorotazionali. Si considerano vari modelli per la produzione degli impulsi angolari. La tendenza dei momenti d'inerzia risultante sperimentalmente è riprodotta da un modello in cui solo i nucleoni esterni a un determinato « core » producono l'impulso angolare totale. La frequenza vibrazionale non è influenzata da tale modello. Si tiene anche conto degli effetti superficiali e dell'influenza delle forze dipendenti dalla velocità.

(*) *Traduzione a cura della Redazione.*

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Gauge Invariance of Baryon Equation.

C. Ceolin and T. Toyoda (*)

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 22 Aprile 1959)

Summary. — A non local, but microcausal, interaction is written between the pseudoscalar pionic field and the baryonic field ($\mathcal{N}$-Ξ). This interaction is obtained by the elimination of a field Φ_μ (associated to each spinor field by the requirement that the theory be not only « Lorentz covariant » but « affine covariant ») through which the π-meson is coupled to the baryons ($\mathcal{N}$-Ξ).

Introduction.

It was recently shown (1) that it is possible to write an equation of Gürsey's type where to 8-component spinor represents either the proton and the Ξ^- or the neutron and Ξ^0. Assuming that the theory is « affine covariant » as well as « Lorentz covariant », we have seen that it possible to associate a neutral vector or pseudovector field Φ_μ to each spinor field and deduce thus from Gürsey's equation two fundamental conservation laws namely the electric charge conservation law and the baryonic number conservation law.

The interest of the present work is to show that, should a pseudo-scalar field, (π-meson) be coupled with the baryonic field by means of the Φ_μ field only, by eliminating such field it is possible to obtain a non-local interaction being however micro-causal.

In Section **1** the main features of the generalized Dirac equation are summarized. In Section **2** a Lagrangian is obtained which leads to the Dirac equation and by eliminating the intermediate field we obtain a non-local interaction. In Section **3** the possibility is considered that the pseudovector neutral field Φ_μ be composed by two different spinor fields.

(*) On leave of absence from Rikkyo University, Tokyo.
(1) T. Toyoda: *Nucl. Phys.*, **8**, 661 (1958).

1. - Generalized Dirac equation.

Making use of covariant formalism for the conservation law of baryon numbers, we have proposed the 8-component Dirac equation which is necessarily associated with vector field interaction [1]

$$(1)\qquad \begin{pmatrix} \gamma^\mu(\delta_\mu + i(\varepsilon + (1+\varepsilon)\gamma_5)\Phi_\mu) & im\gamma_5 \\ -im\gamma_5 & \gamma^\mu(\partial_\mu + i(\varepsilon - (1-\varepsilon)\gamma_5)\Phi_\mu) \end{pmatrix} \begin{pmatrix} \Psi \\ \widehat{\Psi} \end{pmatrix} = 0$$

where m is a bare mass of baryon and γ^μ's are usual 4-component Dirac matrices. Φ_μ is a characteristic vector field with which any spinor field satisfying particle-number conservation law is associated. Eq. (1) is obviously invariant under the following generalized gauge transformation (we shall later refer it to G transformation):

$$(2)\qquad \begin{pmatrix} \Psi \\ \widehat{\Psi} \end{pmatrix}^1 = \begin{pmatrix} \exp\left[i(\varepsilon + (i-\varepsilon)\gamma_5)\vartheta\right] & 0 \\ 0 & \exp\left[i(\varepsilon - (i-\varepsilon)\gamma_5)\vartheta\right] \end{pmatrix} \begin{pmatrix} \Psi \\ \widehat{\Psi} \end{pmatrix}$$

and

$$(3)\qquad \Phi'_\mu = \Phi_\mu - \partial_\mu\vartheta\,,$$

where ε is a certain constant to indicate the coupling strength of Ψ with Φ_μ and ϑ can be an arbitrary function of world point.

For the sake of writing convenience let us denote the following eight-by-eight matrices:

$$(4)\qquad \Gamma^\mu \equiv \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \qquad \text{for } \mu = 1, 2, 3, 4$$

$$(5)\qquad \Gamma_5 \equiv \begin{pmatrix} -\gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}$$

$$(6)\qquad \Gamma^5 \equiv \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}$$

$$(7)\qquad X \equiv \begin{pmatrix} \Psi \\ \widehat{\Psi} \end{pmatrix}$$

Then we may define the « adjoint » of X as

$$\overline{X} \equiv X^{\dagger}\Gamma^{4} \equiv (\Psi^{\dagger}, \widehat{\Psi}^{\dagger}) \begin{pmatrix} 0 & \gamma^{4} \\ \gamma^{4} & 0 \end{pmatrix} \equiv (\overline{\widehat{\Psi}}, \overline{\Psi}) , \tag{8}$$

where † means Hermitic conjugate. By using these notations one can rewrite Eq. (1) and the transformation (2) as

$$\{\Gamma^{\mu}\partial_{\mu} + i(\varepsilon\Gamma^{\mu} - (1-\varepsilon)\Gamma^{\mu}\Gamma_{5})\Phi_{\mu} + im\Gamma_{5}\} X = 0 , \tag{9}$$

$$X' = \exp[i(\varepsilon - (1-\varepsilon)\Gamma_{5})\vartheta] X . \tag{10}$$

Recently, grouping p and Ξ^{-} or n and Ξ^{0} into the 8-component X instead of the conventional combination (p, n), DALLAPORTA and TOYODA [2] have shown that the interaction terms appearing in Eq. (1) or Eq. (9) play important roles in connection with the « boson » conjugation and the « spinor » conjugation introduced by BUDINI, DALLAPORTA and FONDA [3]. Namely if one replaces $(1-\varepsilon)\Phi_{\mu}$ and $\varepsilon\Phi_{\mu}$ by an electromagnetic field εA_{μ} and a baryonic field $g\Phi_{\mu}$, respectively, one may obtain the same equation as that of DALLAPORTA and TOYODA,

$$(\Gamma^{\mu}\partial_{\mu} + ig\Gamma^{\mu}\Phi_{\mu} - ie\Gamma^{\mu}\Gamma_{5}A_{\mu} + im\Gamma_{5}) X = 0 . \tag{11}$$

If we assume A_{μ} and Φ_{μ} independent fields, G transformation should be modified as follows:

$$X' = \exp[i(g\vartheta_{g} - e\Gamma_{5}\vartheta_{e}] X , \tag{12}$$

$$A'_{\mu} = A_{\mu} - \partial_{\mu}\vartheta_{e} , \qquad \Phi'_{\mu} = \Phi_{\mu} - \partial_{\mu}\vartheta_{g} . \tag{13}$$

It would be useful to summarize the important properties of Eq. (11) for our later discussions:

i) *G invariance.* That is, Eq. (11) is invariant under the gauge transformation (12) and (13).

ii) *Separability.* Eq. (11) can be separated into two 4-component Dirac

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(2) N. DALLAPORTA and T. TOYODA: *Nuovo Cimento*, to be published.
(3) P. BUDINI, N. DALLAPORTA and L. FONDA: *Nuovo Cimento*, **9**, 316 (1958).

equations by Gürsey's transformation (4),

$$\begin{pmatrix} \Psi \\ \hat{\Psi} \end{pmatrix} = \begin{pmatrix} \beta & \beta \\ \alpha & -\alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{14}$$

where

$$\alpha = \tfrac{1}{2}(1+\gamma_5)\,, \qquad \beta = \tfrac{1}{2}(1-\gamma_5)\,. \tag{15}$$

iii) *Conservation of two kinds of current.* The electric current and the baryonic current are unambiguously defined, that is,

$$J^\mu_e = \overline{X}\Gamma^\mu\Gamma_5 X \qquad \text{for electric current,}$$

$$J^\mu_b = \overline{X}\Gamma^\mu X \qquad \text{for baryonic current,}$$

both of which satisfy strictly the continuity equation in the presence of interactions

$$\partial_\mu J^\mu_e = 0\,, \tag{16}$$

$$\partial_\mu J^\mu_b = 0\,. \tag{17}$$

iv) *Invariance with respect to BDF conjugation.* In our representation the boson conjugation and the spinor conjugation (*) are given explicitely as follows:

$$X^B = \begin{pmatrix} & \gamma_5 \\ \gamma_5 & \end{pmatrix} X\,, \qquad U_B A_\mu U_B^{-1} = A^B_\mu\,, \qquad U_B \Phi_\mu U_B^{-1} = \Phi^B_\mu\,, \tag{18}$$

$$X^S = \begin{pmatrix} & \gamma_5 C \\ \gamma_5 C & \end{pmatrix} \overline{X}^T\,, \qquad U_S A_\mu U_S^{-1} = A^S_\mu\,, \qquad U_S \Phi_\mu U_S^{-1} = \Phi^S_\mu\,, \tag{19}$$

where U_B and U_S are the unitary charge conjugation operators and C is the usual for-by-four charge conjugation operator. It should be noted that for A_μ and Φ_μ appearing in Eq. (11) the following assumptions are made:

$$A^B_\mu = -A_\mu\,, \qquad \Phi^B_\mu = \Phi_\mu\,, \tag{20}$$

$$A^S_\mu = A_\mu\,, \qquad \Phi^S_\mu = -\Phi_\mu\,. \tag{21}$$

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(4) F. GÜRSEY: *Nuovo Cimento*, **7**, 411 (1958); B. TOUSCHEK: *Nuovo Cimento*, **5**, 754, 1281 (1957); **8**, 1818 (1958).

(*) We are indebted to Professor N. DALLAPORTA for pointing out to us this possibility.

As far as we are concerned with the electromagnetic field, no other assumption is possible. In fact, if we assume that the baryonic field does not change its sign under the spinor conjugation, the following three types of interaction are compatible with our requirements ii) to iv):

$$ig'\Gamma^\mu\Gamma^5\Phi_\mu^{PV} \qquad \text{pseudo-vector (in the reduced form),}$$

$$ig''\Phi^{PS} \qquad \text{pseudo-scalar (in the reduced form),}$$

$$ig'''\Gamma_5\Phi^{P} \qquad \text{scalar (in the reduced form).}$$

Therefore there is no room for the vector field in the latter assumption. However, for the baryonic field the latter choice would be more plausible as we shall discuss later. Concerning our requirement i) we have to modify (12) when we take the pseudo-vector type, that is

$$(12)' \qquad X' = \exp[i(g\Gamma^5\vartheta_g - e\Gamma_5\vartheta_e)]X .$$

2. – Lagrangian formalism.

In order to maintain the gauge invariance strictly as well as the other conditions ii) to iv), let us assume that the baryon field X interacts directly withthe electromagnetic field A_μ and the baryonic pseudo-vector field Φ_μ, respectively. This can be readily described by a Lagrangian

$$(22) \qquad \mathcal{L}' = \frac{1}{2i}[\overline{X}(\overleftarrow{\partial}_\mu\Gamma^\mu\Gamma_5 A_\mu - ig\Gamma^\mu\Gamma^5\Phi_\mu - im\Gamma_5)X - \overline{X}(\Gamma^\mu\overrightarrow{\partial}_\mu - ie\Gamma^\mu\Gamma_5 A_\mu + ig\Gamma^\mu\Gamma^5\Phi_\mu + im\Gamma_5)X] .$$

Although one may use the well-known gauge invariant free Lagrangian for the electromagnetic field, it would be advisable to apply Stückelberg's method [5] to the baryonic pseudo-vector field. That is to introduce a pseudo-scalar field which has the same bare mass as the pseudo-vector field Φ_μ. A free Lagrangian of these fields is then given as

$$(23) \qquad \mathcal{L}_\Phi^0 = -(\partial_\mu\Phi^\nu\partial^\mu\Phi_\nu + M^2\Phi_\mu\Phi^\mu) - (\partial_\mu\varphi\partial^\mu\varphi + \tfrac{1}{2}M^2\varphi^2) .$$

Without making any modification of the transformation rule of X and Φ_μ one can impose the following transformation rule on the pseudo-scalar field

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[5] E. C. G. STÜCKELBERG: *Helv. Phys. Acta*, **11**, 225, 299 (1938).

and assume the subsidiary condition:

$$\varphi' = \varphi + M\vartheta g\,, \qquad (\Box - M^2)\vartheta g = 0\,, \tag{24}$$

$$(\partial_\mu \Phi^\mu + M\varphi)\Psi = 0\,. \tag{25}$$

In spite of its necessity for gauge invariant formalism the pseudo-scalar field has no physical effect at all as is well known.

Before we proceed to the construction of the total Lagrangian, let us look over the present situation of the elementary particle physics. First place low energy phenomena have been explained fairly well by π-meson theory ([6]). Secondly, it has become clearer that the necessity of a non-local interaction, or a cut-off, is inevitable for a consistent framework of quantum field theory itself. However, one of the most serious difficulties of non-local theory is how theory can connect the non-local non-causal region to our causal world. Thirdly among many conservation laws the electric charge conservation and the baryon number conservation seem to us a supreme physical principle ([8,9]).

Recently BUDINI and FONDA ([10]) have shown that one can derive an effective non-local interaction from the conventional local field theory by making use of an intermediate field in such a way that the micro-causality is maintained troughout the way of derivation. Of course all difficulties of divergence still remain, but they are shifted into the intermediate field which will manifest itself in the very high energy region.

Being led by the above considerations we shall introduce a new pseudo-scalar field which may couple with baryons only through an interaction with Φ_μ so that the interaction is necessarily a derivative coupling. It should be noted that the continuity equation of the baryon number can be derived only from Lagrangian (22) so that any interaction of Φ_μ with other fields which do not interact directly with baryons has no influence on the continuity equation. The interaction Lagrangian is now given as

$$\mathcal{L}_{\text{int}} = -g(\bar{X}\Gamma^\mu\Gamma^5 X)\Phi_\mu - f\Phi_\mu\partial^\mu\Phi + e(\bar{X}\Gamma^\mu\Gamma X)A_\mu\,, \tag{26}$$

where g and f are coupling constants. Since Φ does not couple directly with

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([6]) *Supplement of Progr. Theor. Phys.*, no. 3 (1956) edited by M. TAKETANI; G. F. CHEW: *Proc. of Annual International Conference on High Energy Physics at CERN* (1958), p. 93, edited by B. FERRETTI.

([7]) L. D. LANDAU, A. A. ABRIKOSOV and I. M. HALATNIKOV: *Dokl. Akad. Nauk SSSR*, **95**, 773, 1177 (1954); **96**, 261 (1955).

([8]) E. P. WIGNER: *Proc. Natl. Acad. Sci. U.S.*, **38**, 449 (1952).

([9]) B. FERRETTI: *Nuovo Cimento*, **4**, 151 (1956); **5**, 761 (1957); **6**, 204 (1957).

([10]) P. BUDINI and L. FONDA: *Nuovo Cimento*, **5**, 666 (1957).

baryon field X and is a neutral field, it does not seem necessary to impose any gauge transformation on Φ.

Now one can write down a total Lagrangian

$$\mathcal{L} = \mathcal{L}_X^0 + \mathcal{L}_A^0 + \mathcal{L}_\Phi^0 + \mathcal{L}_\Phi^0 + \mathcal{L}_{\text{int}} , \tag{27}$$

where

$$\mathcal{L}_X^0 = \frac{1}{2i}[\overline{X}(\overleftarrow{\partial}_\mu \Gamma^\mu - im)X - \overline{X}(\Gamma^\mu \overrightarrow{\partial}_\mu + im\Gamma_5)X] , \tag{28}$$

$$\mathcal{L}_A^0 = -\tfrac{1}{2}(\partial_\mu A^\mu - \partial_\nu A^\mu)(\partial^\mu A_\nu - \partial^\nu A_\mu) - (\partial_\mu A^\mu)(\partial^\mu A_\mu) , \tag{29}$$

$$\mathcal{L}_\Phi^0 = -(\partial_\mu \Phi^\nu \partial^\mu \Phi_\nu + M^2 \Phi_\mu \Phi^\mu) - (\partial_\mu \varphi \partial^\mu \varphi + \tfrac{1}{2}M^2 \varphi^2) , \tag{30}$$

$$\mathcal{L}_\varphi^0 = -(\partial_\mu \Phi \partial^\mu \Phi + \tfrac{1}{2}\varkappa^2 \Phi^2) , \tag{31}$$

$$\mathcal{L}_{\text{int}} = -g(\overline{X}\Gamma^\mu \Gamma^5 X)\Phi_\mu - f\Phi_\mu \partial^\mu \Phi + e(\overline{X}\Gamma^\mu \Gamma_5 X)A_\mu , \tag{32}$$

where M and $\varkappa$ are the masses of Φ_μ and Φ respectively.

Applying the variational principle to the total Lagrangian with respect to Φ_μ and Φ one obtains the Klein-Gordon equations with interaction terms as follows:

$$(\Box - M^2)\Phi^\mu = g(\overline{X}\Gamma^\mu \Gamma_5 X) + f\partial^\mu \Phi , \tag{33}$$

$$(\Box - \varkappa^2)\Phi = f\partial_\mu \Phi^\mu . \tag{34}$$

In order to eliminate the intermediate field Φ^μ we shall use Budini-Fonda's method. After simple calculations we obtain

$$(\Box - \varkappa_1^2)\left(\Phi, (x)(= -fg\int \Delta_{M'}^c (x - x')\, \partial_\mu (X\Gamma^\mu \Gamma_5 X)\right) , \tag{35}$$

where

$$\varkappa_1^2 = \tfrac{1}{2}(M^2 + \varkappa^2 - f^2 - \sqrt{(M^2 - \varkappa^2 - f^2)^2 + 4f^2\varkappa^2}) , \tag{36}$$

$$M_1^2 = \tfrac{1}{2}(M^2 + \varkappa^2 - f^2 + \sqrt{(M^2 - \varkappa^2 - f^2)^2 + 4f^2\varkappa^2}) . \tag{37}$$

and Δ_M^c is a causal Green function satisfying

$$(\Box - M_1^2)\Delta_{M_1}^c (x - x') = -\partial^4 (x - x') \tag{38}$$

and having in momentum space the following expression

$$\frac{1}{K^2 + M_1^2 - i\varepsilon} . \tag{39}$$

3. – Discussion.

In our theory it is essential to assume that the field associated with baryons is described by a real function due to the postulate of baryon number conservation (see Ref. [1]). Therefore, it seems difficult to make transition between different charged baryons through such a neutral field even if one tries to introduce another intermediate field. However, this difficulty may be overcome by taking into account the difference of two Φ_μ, that is, one couples with the electromagnetic field and the other does not.

Concerning the assumption that we made for the spinor conjugation of Φ^μ field, WOLFENSTEIN and RAVENHALL's remark should be reminded [11]. In fact, if we suppose that the neutral pseudo-vector field Φ_μ is composed of two different spinor fields as

$$\Phi^\mu = (\overline{N}\gamma^\mu\gamma_5\Lambda) + (\overline{\Lambda}\gamma^\mu\gamma_5 N) + \text{complex conjugate} \tag{39}$$

one can show that Φ^μ does not change its sign under the boson conjugation nor under the spinor conjugation. In order to make clear the physical implication of these conjugation for boson fields we had better rewrite these conjugation in the reduced 4-component representation (see Ref. [2]):

$$\text{boson conjugation} \qquad N \to \Lambda\,, \qquad \Lambda \to \mathrm{N}\,, \tag{40}$$

$$\text{spinor conjugation} \qquad N \to N^c\,, \qquad \Lambda \to -\Lambda^c\,, \tag{41}$$

It is quite evident that Φ^μ is invariant against the boson conjugation. To perform the spinor conjugation of Φ^μ one may just replace N and Λ by N^c and $-\Lambda^c$ respectively, that is,

$$\begin{aligned}(\Phi^\mu)^S &= \overline{N}^c\gamma^\mu\gamma_5(-\Lambda^c) + (-\overline{\Lambda}^c)\gamma^\mu\gamma_5 N^c + \text{c. c.}\\ &= N^T C^{-1}\gamma^\mu\gamma_5 C\overline{\Lambda}^T + \Lambda^T C^{-1}\gamma^\mu\gamma_5 C\overline{N}^T + \text{c. c.}\\ &= (\overline{\Lambda}\gamma^\mu\gamma_5 N)^T + (\overline{N}\gamma^\mu\gamma_5\Lambda)^T + \text{c. c.} = \Phi^\mu = \Phi^{\mu\dagger} = \Phi^{\mu *}\,,\end{aligned}$$

where the well-known formulas ($N^c = C\overline{N}^T$, $\overline{N}^c = -N^T C^{-1}$) are used. As is seen in the above discussion, N and Λ need not be neutral spinors.

Finally, we would like to emphasize that all baryons, no matter charged, neutral, hypercharged, nor any other kind of characterization, couple directly

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[11] L. WOLFENSTEIN and D. G. RAVENNHALL: *Phys. Rev.*, **88**, 279 (1952).

with the neutral pseudo-vector field so that the baryon number conservation law may hold strictly. However, for low energy phenomena this field may be absorbed into the form factor of baryons and may manifest itself in the very high energy region.

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We are very grateful to Professor N. DALLAPORTA and Professor P. BUDINI for their interest in the present work and valuable discussions.

One of us (T.T.) would like to express his sincere thanks to Professor N. DALLAPORTA for his kind hospitality extended to him at the Istituto di Fisica dell'Università, Padova.

RIASSUNTO

Si scrive un'interazione non locale, tuttavia microcausale, fra il campo pseudoscalare pionico e il campo barionico ($\mathcal{N}$-Ξ). Questa interazione si ottiene dall'eliminazione di un campo Φ_μ (associato a ciascun campo spinoriale dalla richiesta che la teoria sia oltre che « Lorentz covariant » anche « affine covariant ») mediante il quale il mesone π è accopiato ai barioni ($\mathcal{N}$-Ξ).

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K^-, $\overline{K}^0$ Relative Parity and the K^--d Charge Exchange Reaction (*).

T. B. Day, G. A. Snow and J. Sucher

Department of Physics, University of Maryland - College Park, Md.

(ricevuto il 27 Aprile 1959)

Summary. — The reaction $K^- + d \to \overline{K}^0 + 2n$ is studied in the hope that limitations on the final states for the two neutrons due to statistics will help differentiate between even and odd relative parity of the K^- and $\overline{K}^0$ mesons. The total cross-section for the reaction is calculated in the impulse approximation, and expressed in terms of the corresponding hydrogen cross-section for both even and odd parity. The calculations are carried out *a*) with the use of antisymmetrized plane waves for the two neutrons, and *b*) in the closure approximation. The ratio of the odd to even parity cross-sections is about a factor of two in either approximation. The total cross-section for odd parity in the plane wave approximation is found to be about 4 mb at a K^- laboratory momentum of 136 MeV/c.

1. – Introduction.

The original Gell-Mann [1]-Nishijima [2] classification scheme of strange particles, which has proved so successful, treats the K^- and $\overline{K}^0$ mesons (and their charge conjugate particles, K^+ and K^0) as a doublet in isotopic spin space. The assumption of charge independence for the strong interactions would then imply that the K^- and $\overline{K}^0$ have the same intrinsic parity as manifested

(*) This research was supported in part by the U.S. Atomic Energy Commission and by the U.S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

(1) Gell-Mann: *Phys. Rev.*, **92**, 833 (1953).

(2) K. Nishijima: *Prog. Theor. Phys.* (*Kyoto*), **13**, 285 (1955).

through their strong interactions. Nevertheless, in a search for deeper symmetry relationships, PAIS [3] has raised the question as to whether the relative parity, $p(K)$, of the K^- and $\overline{K}^0$ mesons, is in fact odd or even. Indeed, the energy and angular distributions of some associated production reactions have been cited [4] as possible evidence for odd $p(K)$.

The purposes of this note is to examine the feasibility of determining $p(K)$ from a comparison of the $(K^-, \overline{K}^0)$ charge exchange cross-sections on protons and deuterons. The basic argument, which has been briefly discussed by L. B. OKUN [5], is exactly analogous to the well-known discussions of the relative parity of π^- and π^0 mesons as deduced from the corresponding (π^-, π^0) charge exchange reactions [6]. The recent result of ROSENFELD *et al.* [7] that the mass of the $\overline{K}^0$ meson is larger than the mass of the K^- meson by (3.9 ± 0.6) MeV is incorporated in the calculations that follow.

Let us consider the $(K^-, \overline{K}^0)$ charge exchange cross-sections in hydrogen and deuterium near threshold; that is, the reactions

$$K^- + p \to n + \overline{K}^0 , \tag{1}$$

and

$$K^- + d \to n + n + \overline{K}^0 . \tag{2}$$

It is the restriction imposed by the Pauli principle on the wave function of the (n, n) system in reaction (2) that makes the cross-section sensitive to the value of $p(K)$. Table I shows the allowed final states for the $n+n+\overline{K}^0$ system for incident K^--mesons in either $l = 0$ or $l = 1$ states of relative orbital angular momentum.

Since the reaction is endothermic, the threshold dependence of the cross-section on q_m, the maximum $\overline{K}^0$ momentum in the center of mass system, is as shown in Table I. The essential point is then that the charge exchange cross-section in deuterium is expected to be larger for $p(K)$ odd than for $p(K)$ even due to the extra factor of q_m^2 present in the latter case. In constructing Table I, and throughout this paper, we assume that parity is conserved in the $(K^-, \overline{K}^0)$ charge exchange process.

To make the comparison between even or odd $p(K)$ more quantitative, we shall use the impulse approximation to calculate the cross-section for re-

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(3) A. PAIS: *Phys. Rev.*, **112**, 624 (1958); *Phys. Rev. Lett.*, **1**, 418 (1958).
(4) J. G. TAYLOR: *Nucl. Phys.*, **9**, 357 (1959).
(5) L. B. OKUN: *Journ. Exper. Theor. Phys.* **30**, 218 (1956).
(6) See for example R. MARSHAK: *Rev. Mod. Phys.*, **23**, 137 (1951).
(7) A. H. ROSENFELD, F. T. SOLMITZ and R. D. TRIPP: *Phys. Rev. Let.*, **2**, 110 (1959); see also F. S. CRAWFORD, M. CRESTI, M. L. GOOD, M. L. STEVENSON and H. K. TICHO: *Phys. Rev. Lett.*, **2**, 112 (1959).

TABLE I. – *Angular momentum analysis of* $K^- + d \to n + n + \overline{K}^0$ *near threshold.*

Initial state (*)	Final state (*)		$\overline{K}^0$ momentum dependence near threshold (**)	
	(i) p(K) even	(ii) p(K) odd	(i) p(K) even	(ii) p(K) odd
$^3S_1(s)$	$^3P_{0,1,2}(p)$	$^1S_0(p)$	q_m^8	q_m^8
		$^3P_1(s)$		q_m^6
$^3S_1(p)$	$^1S_0(p)$	$^1S_0(s)$	q_m^8	q_m^4
	$^3P_{0,1,2}(s)$	$^3P_{0,1,2}(p)$	q_m^6	q_m^8

(*) The usual spectroscopic notation is employed, *e.g.*, $^3S_1(s)$ denotes an $l=0$ K^--meson incident on the deuteron which is taken to be in a $^3S_1$ state; $^3P_0(p)$ denotes the final state with 2 neutrons in a $^3P_0$ state and the $\overline{K}^0$-meson in an $l=1$ state relative to the two neutrons.

(**) Here q_m is the maximum momentum of the $\overline{K}^0$ in the CM system.

action (2). This approximation has been used by many authors [8]. The calculations by FERNBACH *et al.* [9] and ROCKMORE [10] on the cross-sections for π-mesons on deuterium most closely resemble the calculations presented here. As will be seen, the simple q_m dependences given in Table I hold, unfortunately, only over a negligibly small energy interval above threshold. Nevertheless, the qualitative point persists in that the cross-section for reaction (2) relative to reaction (1) is larger for p(K) odd than for p(K) even over a considerable energy region.

In Section **2** the impulse approximation calculation is described; the wave functions for the outgoing particles are taken to be plane waves. Section **3** deals with the closure approximation. The results are summarized and discussed in Section **4**.

2. – Plane wave calculation.

The total cross-section for reaction (2) can be written as

$$\sigma_d = \frac{2\pi}{v_K + v_d} \frac{1}{3} \sum_{\substack{i,f \\ \text{spins}}} \int \frac{d^3\boldsymbol{K}}{(2\pi)^6} d^3\boldsymbol{q}\, \delta(E_p - E_i) |M|^2, \tag{3}$$

where $\boldsymbol{K}$ is the relative momentum of the two neutrons; $\boldsymbol{q}$ is the momentum of the $\overline{K}^0$-meson in the CM system; the sum in Eq. (3) is over initial and final

(8) G. F. CHEW: *Phys. Rev.*, **80**, 196 (1950); K. M. WATSON and R. N. STUART: *Phys. Rev.*, **82**, 738 (1951); G. F. CHEW and H. W. LEWIS: *Phys. Rev.*, **84**, 779 (1951); M. LAX and H. FESHBACH: *Phys. Rev.*, **88**, 509 (1952).

(9) S. FERNBACH, T. A. GREEN and K. M. WATSON: *Phys. Rev.*, **84**, 1084 (1951).

(10) R. M. ROCKMORE: *Phys. Rev.*, **105**, 256 (1957).

spin states; v_K and v_d are the velocities of the K^- and d, respectively, in the CM system; M is the matrix element for the transition. We denote the CM momentum of the K^--meson by $\boldsymbol{k}$.

In the impulse approximation

$$M=\int d\boldsymbol{r}\, \exp\left[-i\,\frac{\boldsymbol{q}}{2}\cdot\boldsymbol{r}\right] v^*_{nn}(\boldsymbol{r}) t \exp\left[i\boldsymbol{k}\cdot\frac{\boldsymbol{r}}{2}\right] u_d(\boldsymbol{r})\,, \tag{4}$$

where $u_d(\boldsymbol{r})$ and $v_{nn}(\boldsymbol{r})$ are the wave functions for the deuteron and the two neutrons (spin indices are suppressed), and t is the transition operator for the reaction $K^- + p \to \overline{K}^0 + n$.

In the case of even $p(K)$, t must be a scalar. The general form of t is then

$$t = a + a'\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{q}'\,, \tag{5}$$

where $\boldsymbol{k}'$ and $\boldsymbol{q}'$ are the K^- and $\overline{K}^0$ momenta in the K^--p rest system, and a and a' are scalars. Since we are only interested in the cross-section near threshold, it is reasonable to take

$$t = a = \text{constant}\,. \tag{6}$$

An examination of the preliminary data on reaction (1) of EBERHARD *et al.* [11] indicates that this assumption is invalid for K^- laboratory momenta above ~ 150 MeV/c. For the case of odd $p(K)$, t must be a pseudoscalar, of the general form

$$t = b\,\boldsymbol{\sigma}\cdot\boldsymbol{k}' + b'\boldsymbol{\sigma}\cdot\boldsymbol{q}'\,. \tag{7}$$

In a similar spirit to the above, near threshold we take

$$t = b\,\boldsymbol{\sigma}\cdot\boldsymbol{k}'\,, \tag{8}$$

with b a constant. With these approximations, the cross-sections for reaction (1) are

$$\sigma^e_H = \mu_{K^-p}\mu_{\overline{K}^0 n}|a|^2 q'/\pi k'\,, \tag{9}$$

and

$$\sigma^0_H = \mu_{K^-p}\mu_{\overline{K}^0 n}|b|^2 q' k'/\pi\,, \tag{10}$$

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[11] P. EBERHARD, A. H. ROSENFELD, F. T. SOLMITZ, R. D. TRIPP and M. B. WATSON: *Phys. Rev. Lett.*, **2**, 312 (1959).

for $p(\mathrm{K})$ even and odd respectively. Here $\mu_{\mathrm{K}^-\mathrm{p}}$ and $\mu_{\bar{\mathrm{K}}^0\mathrm{n}}$ are the reduced masses of the $\mathrm{K}^-\mathrm{p}$ and $\bar{\mathrm{K}}^0\mathrm{n}$ systems respectively.

In evaluating M, we take for the spatial wave function of the deuteron

$$u_{\mathrm{d}}(\boldsymbol{r}) = N(\exp[-\alpha r] - \exp[-\beta r])/r\,, \tag{11}$$

where

$$\begin{cases} N^2 = \alpha/2\pi(1-\alpha\varrho)\,, \\ \alpha \;\; = (M\varepsilon)^{\frac{1}{2}} = 45.6\ \mathrm{MeV/c}\,, \\ \varepsilon \;\; = \text{deuteron binding energy}\,, \\ \varrho \;\; = 4(\alpha+\beta)^{-1} - \beta^{-1} = 1.70\cdot 10^{-13}\ \mathrm{cm}, \end{cases} \tag{12}$$

and

$$\beta = 6.2\,\alpha\,.$$

2·1. $p(\mathrm{K})$ *even.* – Using the approximation of Eq. (6) for the t operator, we obtain from Eq. (4)

$$\frac{1}{3}\sum_{i,f}|M|^2 = \left|\frac{4\pi N a}{\sqrt{2}}\,F_-(\boldsymbol{k};\boldsymbol{q},\boldsymbol{K})\right|^2, \tag{13}$$

where

$$F_\pm = \left\{\left[\alpha^2+\left(\boldsymbol{K}+\frac{\boldsymbol{q}-\boldsymbol{k}}{2}\right)^2\right]^{-1} \pm \left[\alpha^2+\left(-\boldsymbol{K}+\frac{\boldsymbol{q}-\boldsymbol{k}}{2}\right)^2\right]^{-1} + \right. \tag{14}$$

$$\left. + \left[\beta^2+\left(\boldsymbol{K}+\frac{\boldsymbol{q}-\boldsymbol{k}}{2}\right)^2\right]^{-1} \pm \left[\beta^2+\left(-\boldsymbol{K}+\frac{\boldsymbol{q}-\boldsymbol{k}}{2}\right)^2\right]^{-1}\right\}.$$

Since in our approximation t is spin-independent, only the spin-triplet (2n) system contributes to equation (4). Hence

$$v_{\mathrm{nn}}(\boldsymbol{r}) = (\exp[i\boldsymbol{K}\cdot\boldsymbol{r}] - \exp[-i\boldsymbol{K}\cdot\boldsymbol{r}])/\sqrt{2}\,. \tag{15}$$

In the notation previously introduced, energy conservation takes the form

$$\frac{k^2}{2\mu_{\mathrm{K}^-,\mathrm{d}}} = \frac{q^2}{2\mu_{\bar{\mathrm{K}}^0,\,2\mathrm{n}}} + \frac{K^2}{M} + Q\,, \tag{16}$$

where

$$Q = m_{\bar{\mathrm{K}}^0} - m_{\mathrm{K}^-} + m_{\mathrm{n}} - m_{\mathrm{p}} + \varepsilon \cong 7.1\ \mathrm{MeV}\,,$$

the μ's are reduced masses, and M is the nucleon mass.

The integration over the magnitude of $\boldsymbol{K}$ in Eq. (3) gives

$$\sigma_{\mathrm{d}}^{e} = \frac{2\pi\mu_{\mathrm{K}^-,\mathrm{D}}}{k}\,\frac{(4\pi N)^2\,|a|^2}{2}\,\frac{M}{2}\left(\frac{M}{2\mu_{\overline{\mathrm{K}}^0,2\mathrm{n}}}\right)^{\frac{1}{2}}\frac{1}{(2\pi)^6}\,I_-\,, \tag{18}$$

where

$$I_\pm = \int_0^{q_m} (q_m^2 - q^2)^{\frac{1}{2}} q^2\,\mathrm{d}q\,\mathrm{d}\Omega_q\,\mathrm{d}\Omega_K\,|F_\pm(\boldsymbol{k};\,\boldsymbol{q},\,\boldsymbol{K})|^2, \tag{19}$$

and

$$q_m^2 = \frac{\mu_{\overline{\mathrm{K}}^0,2\mathrm{n}}}{\mu_{\mathrm{K}^-,\mathrm{d}}}\,(k^2 - k_0^2) \cong (k^2 - k_0^2)\,. \tag{20}$$

k_0 is the CM threshold momentum,

$$k_0 = (2\mu_{\mathrm{K}^-\mathrm{d}}\,Q)^{\frac{1}{2}} \cong 76.1\ \mathrm{MeV/c}\,. \tag{21}$$

In $F_-(\boldsymbol{k};\,\boldsymbol{q},\,\boldsymbol{K})$, K is now given by

$$K^2 = \frac{M}{2\mu_{\overline{\mathrm{K}}^0,2\mathrm{n}}}\,(q_m^2 - q^2) \simeq \frac{5}{4}\,(q_m^2 - q^2)\,. \tag{22}$$

The evaluation of I_- is given in Appendix A. It is estimated that this evaluation is accurate to about 10%. It is convenient to express $I_\pm$ in the form

$$I_\pm = I_1 \pm I_2\,, \tag{23}$$

where I_1 and I_2 are defined by Eqs. (A.8) of the Appendix.

Fig. 1 presents I_1 and I_2 plotted against η^2 where

$$\eta = k/k_0\,. \tag{24}$$

The dependence of I_- on q_m very near threshold is given by (see Eq. (A.16))

$$I_- \xrightarrow[q_m \to 0]{} \frac{100\pi}{3}\,\frac{(4\pi)^2}{[1 + 4(\alpha/k_0)^2]^4}\left(\frac{q_m}{k_0}\right)^6. \tag{25}$$

Comparison of this limiting behavior with the more accurate I_- obtained from Eq. (23) and Fig. 1, indicates that this approximation is poor for $\eta^2 \geqslant 1.05$.

2·2. p(K) *odd.* – In this case, the appropriate transition operator, t, is given by Eq. (8). k', the relative momentum of the K^--meson and proton in their rest system, is related to k, the relative momentum of the K^--meson

and deuteron in their rest system, by

$$\boldsymbol{k}' = \left(\frac{M + \frac{1}{2}m_{K^-}}{M + m}\right)\boldsymbol{k} + \left(\frac{m_{K^-}}{M + m_{K^-}}\right)\boldsymbol{d}\,, \tag{26}$$

where $\boldsymbol{d}$ is the neutron-proton relative momentum in the K^--d rest system. For simplicity the $\boldsymbol{d}$ term will be neglected in this calculation ([12]).

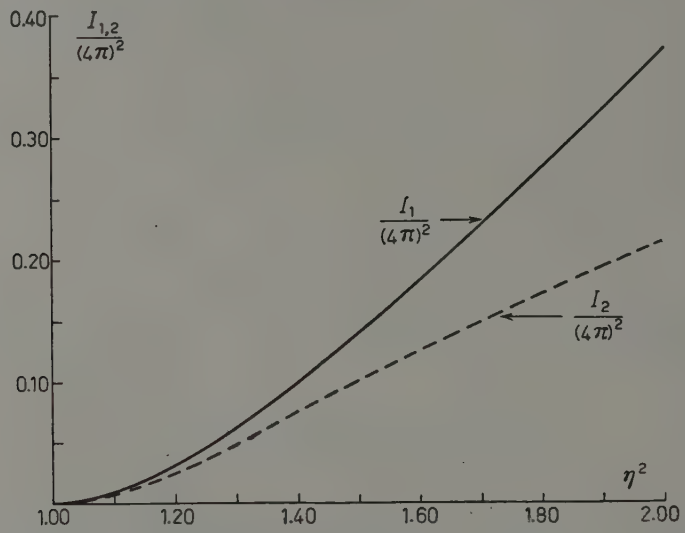

Fig. 1. – Plane-wave computation. The integrals I_1 and I_2, defined in Eq. (A.8) are plotted as a function of $\eta^2 = (k/k_0)^2$. ($k = CM$ momentum of the K^-; k_0 = the threshold value of k).

On substituting into Eq. (4), and performing the spin sum, we obtain

$$\frac{1}{3}\sum_{i,f}|M|^2 = \left(\frac{M + \frac{1}{2}m_{K^-}}{M + m_{K^-}}\right)^2 |b|^2 k^2 \frac{(4\pi N)^2}{2}\left(\frac{2}{3}F_-(\boldsymbol{k};\boldsymbol{q},\boldsymbol{K}) + \frac{1}{3}F_+(\boldsymbol{k};\boldsymbol{q},\boldsymbol{K})\right), \tag{27}$$

where, as before, F_- corresponds to transitions to a spin-triplet two-neutron state, and F_+ corresponds to a spin-singlet two-neutron final state. The cross-section can then be written in the form

$$\sigma_d^0 = \frac{2\pi\mu_{K^-d}}{k}\,\frac{(4\pi N)^2}{2}\,\frac{M}{2}\left(\frac{M}{2\mu_{\bar{K}^0,2n}}\right)^{\frac{1}{2}} |b|^2 k^2\,\frac{1}{(2\pi)^6}\left(\frac{M + \frac{1}{2}m_{K^-}}{M + m_{K^-}}\right)^2\left(\frac{2}{3}I_- + \frac{1}{3}I_+\right), \tag{28}$$

where I_+ and I_- are defined in Eq. (17). Thus, from Eqs. (16), (21) and (26)

([12]) The d term is smaller than the k term by a factor $\sim (\alpha/3k)$ which is of the order of 0.15.

it is seen that the essential effect of the Pauli principle in going from even $p(K)$ to odd $p(K)$ is to replace $I_1 - I_2$ by $I_1 - \frac{1}{3}I_2$.

Considerations about threshold behavior similar to those in the even parity case (see Eq. (A.17)) lead to

$$I_+ \xrightarrow[q_m \to 0]{} \frac{4\pi\,(4\pi)^2}{[1+4(\alpha/k_0)^2]^2}\left(\frac{q_m}{k_0}\right)^4. \tag{29}$$

Again, this approximation is good only for $\eta^2 \leqslant 1.05$.

3. – Closure approximation.

The calculation of the K^--d cross-section within the framework of the closure approximation is completely analogous to the corresponding calculation in the π^--d problem ([9,10]). One finds (with a and b the same constants as in Section 2),

$$\sigma_d^s = \frac{\mu_{K^-d}}{\pi k}|a|^2\left(\frac{q^2\,dq}{dE}\right)_{av}(1-\Delta)\,, \tag{30}$$

$$\sigma_d^0 = \frac{\mu_{K^-d}}{\pi k}|b|^2\left(\frac{M+\frac{1}{2}m_{K^-}}{M+m_{K^-}}\right)^2 k^2\left(\frac{q^2\,dq}{dE}\right)_{av}\left(1-\frac{\Delta}{3}\right), \tag{31}$$

where

$$\Delta \equiv \int\frac{d\Omega_d}{4\pi}\int d^3\boldsymbol{r}\,d^3\boldsymbol{r}'\,\delta(\boldsymbol{r}+\boldsymbol{r}')\,u_d^*(\boldsymbol{r}')\,u_d(\boldsymbol{r})\,\exp\left[i\frac{(\boldsymbol{k}-\boldsymbol{q})}{2}\cdot(\boldsymbol{r}-\boldsymbol{r}')\right]. \tag{32}$$

In Eq. (31), the factor $1-\Delta/3$ is the result of adding $\frac{2}{3}(1-\Delta)$ and $\frac{1}{3}(1+\Delta)$ (Compare with Eq. (28)).

On integration, Eq. (32) gives

$$\Delta = \frac{4\pi N^2}{kq_{av}}\left[g(k,\,q_{av};\,\alpha)+g(k,\,q_{av};\,\beta)-2g\left(k,\,q_{av};\,\frac{\alpha+\beta}{2}\right)\right], \tag{33}$$

where

$$g(k,\,q;\,\gamma) = \frac{k}{2}\,\mathrm{tg}^{-1}\left[\frac{4\gamma q}{4\gamma^2+(k^2-q^2)}\right]+\frac{q}{2}\,\mathrm{tg}^{-1}\left[\frac{4\gamma k}{4\gamma^2-(k^2-q^2)}\right]- \tag{34}$$

$$-\frac{\gamma}{2}\ln\left[\frac{4\gamma^2+(k+q)^2}{4\gamma^2+(k-q)^2}\right].$$

In the closure approximation the restriction of energy conservation, Eq. (16), on the variables q and K is ignored, the K integration is extended to infinity, and $q = q(K)$ is replaced by a suitable « average » value, q_{av}. From the expected form of the $\overline{K}^0$ momentum spectrum, q_{av} should lie between $\frac{2}{3}q_m$ and q_m.

It is seen that taking q_{av}/q_m to be a constant, independent of k, yields a threshold dependence linear in q_m for σ_d^e and σ_d^0 instead of the correct q_m^6 and q_m^4 dependence respectively.

This completely erroneous result of the closure approximation very near threshold was to be expected, since in this region the extension of the K integration to infinity grossly overestimates the cross-section. On the other hand, for the highest K^- momentum considered here ($k_{lab} = 136$ MeV/c), a crude estimate indicates that the closure approximation may be in error by about 50% due to the extension of the K integration to infinity.

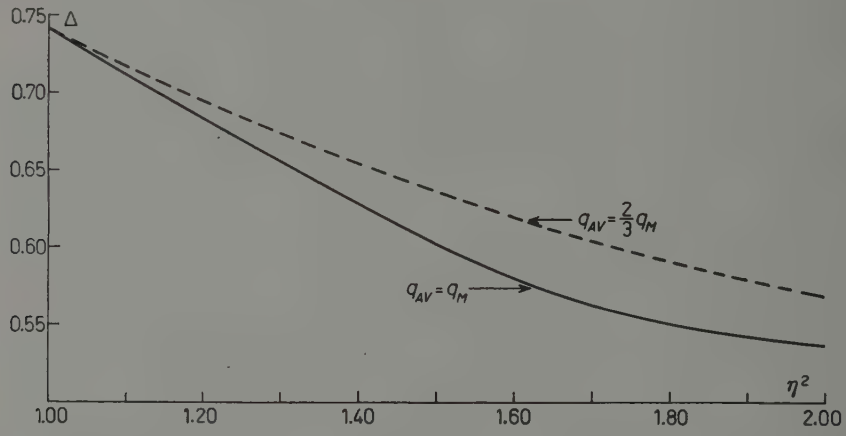

Fig. 2. – Closure computation. Δ versus $\eta^2 \cdot \Delta$ is given by Eq. (32). The deuteron cross-section for $p(K)$ even and odd are proportional to $1 - \Delta$ and $1 - \Delta/3$ respectively (see text for definitions of q_{av} and q_m).

Fig. 2 presents curves of Δ versus η^2 for two choices of q_{av}, $q_{av} = q_m$ and $q_{av} = \frac{2}{3}q_m$. It is seen that Δ is rather insensitive to q_{av} so that the major dependence on q_{av} of the cross-sections in the closure arises from the factor $(q^2\,\mathrm{d}q/\mathrm{d}E)_{av}$ in Eqs. (30) and (31).

4. – Summary and discussion.

By use of Eqs. (9), (10), (18) and (28), the deuteron cross-section may be expressed in terms of the hydrogen cross-section for both even and odd relative parity ([13]):

$$(35)\quad \sigma_d^e(k) = \left[\sigma_H(k')\frac{k'}{q'}\right]\left(\frac{\alpha}{k}\right)\left(\frac{2}{\pi(1-\alpha\varrho)}\right)\left(\frac{M}{2\mu_{\bar{K}^0,2n}}\right)^{\frac{1}{2}}\left(\frac{M}{2\mu_{\bar{K}^0 n}}\right)\left(\frac{\mu_{K^-d}}{\mu_{K^-p}}\right)\frac{1}{(4\pi)^2}(I_1 - I_2)\,,$$

([13]) The differential cross-sections as functions of $\bar{K}_0$ momenta may be inferred from the integral of Eqs. (A.9), (A.10) and (A.11), and from Eqs. (18), (28) and (A.1).

$$(36)\quad \sigma_d^0(k) = \left[\sigma_H(k')\frac{k'}{q'}\right]\left(\frac{k}{k'}\right)^2\left(\frac{M+(m_{K^-}/2)}{M+m_{K^-}}\right)^2 \frac{\alpha}{k}\frac{2}{\pi(1-\alpha\varrho)}\left(\frac{M}{2\mu_{\overline{K}^0,2n}}\right)^{\frac{1}{2}}\left(\frac{M}{2\mu_{\overline{K}^0 n}}\right)\cdot$$
$$\cdot\left(\frac{\mu_{K^-d}}{\mu_{K^-p}}\right)\frac{1}{(4\pi)^2}\left(I_1-\frac{1}{3}I_2\right).$$

Fig. 3 presents σ_d^e and σ_d^0 versus k_L, the K^- momentum in the laboratory system. The values of I_1 and I_2 are taken from Fig. 1. σ_H was taken from reference [11] as

$$(37)\quad \sigma_H \cong 11 \text{ mb} \quad \text{at} \quad k_L' = 130 \text{ MeV/c}.$$

The experimental uncertainty in this value is very large ($\sim 50\%$). When a more accurate value for σ_H at this momentum is obtained, σ_d^e and σ_d^0 of Fig. 3 should be multiplied by $(\sigma_H/11)$.

Examination of these curves shows that the cross-sections differ by a factor of five or more, corresponding to the q_m^6 and q_m^4 behavior in the $p(K)$ even and odd cases, only at momenta too close to threshold to be of practical interest. However, for values of k_L of about 130 MeV/c, where the cross-sections are in the millibar region, the two cases still differ by a factor of about two.

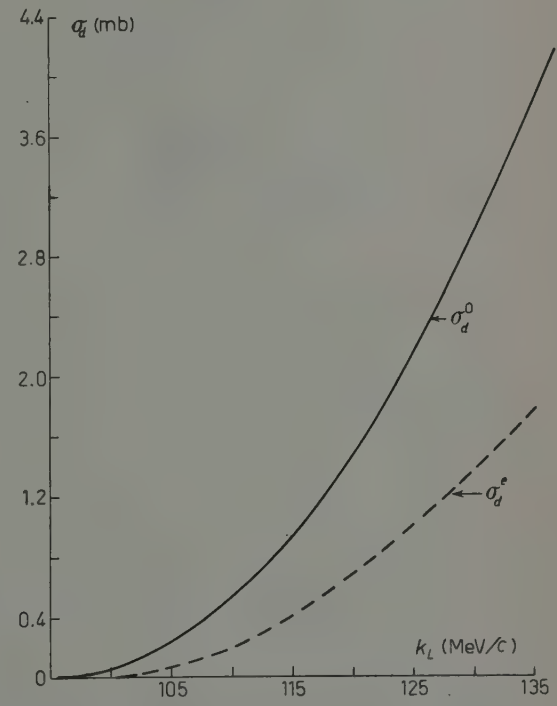

Fig. 3. – Plane-wave computation. The deuteron cross-sections, σ_d^0 and σ_d^e, are given for $p(K)$ odd and $p(K)$ even, respectively, as a function of k_L, the laboratory momentum of the K^--meson. (In computing these cross-sections the K^--p charge exchange cross-section has been taken to be 11 mb at $k_L = 136$ MeV/c [11].

As described in Section 2 and Appendix A, the plane-wave calculations have only been treated approximately. The over-all error arising from these approximations is estimated to be of the order of 20%.

Over and above these numerical approximations, the actual cross-sections may differ considerably from those calculated here due to the fact that the impulse plane-wave approximation neglects several processes which can contribute. The most obvious neglect, and also the one most amenable to cor-

rection, is that of the interaction of the particles in the final state, particularly the two neutrons. In addition, there are three-body effects in the intermediate states including the neutrons, which have been neglected. The two-neutron final state interaction is expected to significantly increase the cross-section only in the case of p(K) odd, since only in that case do space-symmetric two-neutron states appear. For p(K) even, the choice of a spin-independent t operator (see Section 2) implies only odd angular momentum states for the two-neutron system, and these states are known to have weak interactions at small relative momentum.

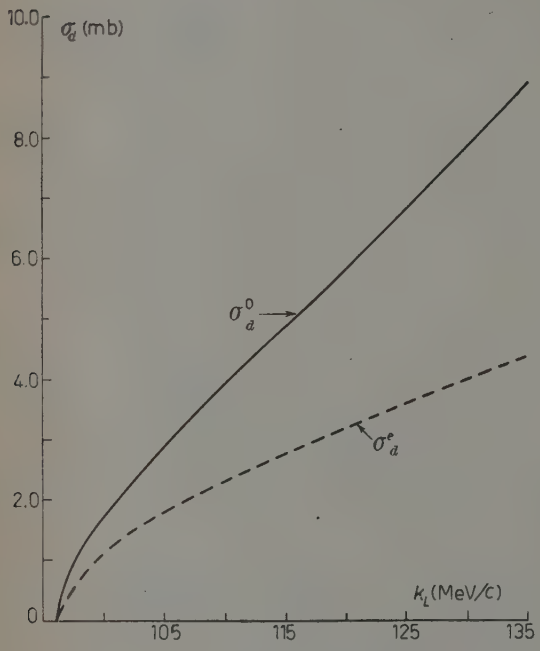

Fig. 4. – Closure computation. σ^0_d and σ^e_d versus k_L. (See caption for Fig. 3) q_{av} was chosen as $\frac{2}{3}\, q_m$.

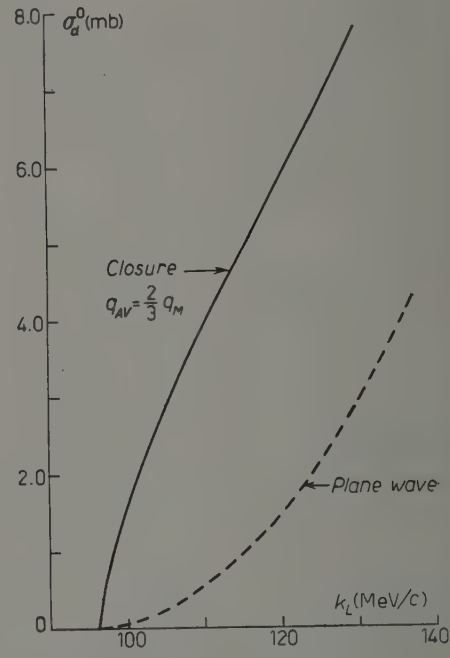

Fig. 5. – Comparison of plane wave and closure ($p_{av} = \frac{2}{3}\, q_m$) computations for the case of odd p(K).

Additional information about the effect of the two-neutron final state interaction may be gained from the results of the closure approximation. Fig. 4 presents σ^e_d and σ^0_d versus k_L in this approximation for the case $q_{av} = \frac{2}{3} q_m$. It is seen that in this approximation also, σ^0_d/σ^e_d is about a factor of 2 for $k_L \sim 130$ MeV/c. More important, one might expect the closure cross-sections to be an upper bound on the true cross-sections in both cases, and in the case of odd p(K) to take into account to some extent a strong final state interaction. In the case of even p(K), one may expect the plane wave calculation to be quite adequate. (Coulomb corrections to the incoming K^--wave function

were estimated, and found to be not important in the momentum range considered here.)

Fig. 5 displays the results of the closure ($q_{av} = \frac{2}{3} q_m$) and plane wave approximations in the case of $p(K)$ odd. If an experimental determination of the cross-section for charge exchange scattering of K^--mesons on deuterons should give a value falling between these two curves, this would be considered as evidence for the relative parity of the $\overline{K}^0$ and K^--mesons to be odd. Should the result fall much below the plane wave calculation for the odd $p(K)$ case, an even relative parity would be indicated.

In conclusion, accurate measurements of the K^--p and K^--d total charge exchange cross-sections, near threshold, offer some hope for determination of the relative parity of K^- and $\overline{K}^0$ mesons.

Appendix A

In evaluating the I_+ of Eq. (19), we neglect the β-terms of $F_\pm$ since $\beta = 6.2\alpha$. Then

$$I_\pm = \int_0^{q_m} (q_m^2 - q^2)^{\frac{1}{2}} q^2 \, dq \, J_\pm , \tag{A.1}$$

where

$$J_\pm = \int d\Omega_q \, d\Omega_K \left| \left[\alpha^2 + \left(\boldsymbol{K} + \frac{\boldsymbol{q} - \boldsymbol{k}}{2} \right)^2 \right]^{-1} \pm \left[\alpha^2 + \left(-\boldsymbol{K} + \frac{\boldsymbol{q} - \boldsymbol{k}}{2} \right)^2 \right]^{-1} \right|^2 . \tag{A.2}$$

If we put

$$J_\pm = J_1 \pm J_2 , \tag{A.3}$$

(where J_1 corresponds to the sum of the squares and J_2 to the cross term in the integrand), J_1 and J_2 can be integrated to yield

$$J_1 = \frac{16\pi^2}{\alpha q k K} \operatorname{tg}^{-1} \left\{ \frac{32 k q \alpha K}{64\alpha^2 K^2 + [(k^2 - q^2) + 4(\alpha^2 - K^2)^2 + 16 q^2 (\alpha^2 - K^2)]} \right\} , \tag{A.4}$$

$$J_2 = \frac{16\pi^2}{2qkK(\alpha^2 + K^2)^{\frac{1}{2}}} \int_{\theta_1}^{\theta_2} dx \ln \left\{ \frac{1 + [K^2/(\alpha^2 + K^2)]^{\frac{1}{2}} \sin x}{1 - [K^2/(\alpha^2 + K^2)]^{\frac{1}{2}} \sin x} \right\} , \tag{A.5}$$

where

$$\theta_1 = 2 \operatorname{tg}^{-1} \left[\frac{k - q}{2(\alpha^2 + K^2)^{\frac{1}{2}}} \right] , \tag{A.6}$$

$$\theta_2 = 2 \operatorname{tg}^{-1} \left[\frac{k + q}{2(\alpha^2 + K^2)^{\frac{1}{2}}} \right] . \tag{A.6}$$

J_2 is related to combinations of Spence functions, but for our purposes, we expand the logarithm in powers of $\sin x$:

$$J_2 = J_{2,0} + J_{2,1} + \ldots \tag{A.7}$$

(where $J_{2,\nu}$ corresponds to $(\sin x)^{2\nu+1}$), and integrate term by term.

Re-writing J_1, doing the integrations for $J_{2,0}$ and $J_{2,1}$, and defining

$$I_1 = \int_0^{q_m} (q_m^2 - q^2)^{\frac{1}{2}} q^2 \, \mathrm{d}q J_1 , \qquad I_2 = J_{2,0} + I_{2,1} = \int_0^{q_m} (q_m^2 - q^2)^{\frac{1}{2}} q^2 \, \mathrm{d}q (J_{2,0} + J_{2,1}) , \tag{A.8}$$

we get

$$\frac{I_1}{(4\pi)^2} = \int_0^{q_m^2} \mathrm{d}y \, \frac{1}{\sqrt{5}\,\alpha k} \, \mathrm{tg}^{-1} \frac{16\sqrt{5}\,\alpha k [y(q_m^2 - y)]^{\frac{1}{2}}}{Vy^2 + Wy + Z} , \tag{A.9}$$

$$\frac{I_{2,0}}{(4\pi)^2} = \int_0^{q_m^2} \mathrm{d}y \, \frac{[y(q_m^2 - y)]^{\frac{1}{2}}}{y^2 - Ry + T} , \tag{A.10}$$

and

$$\frac{I_{2,1}}{(4\pi)^2} = \frac{5}{12} \int_0^{q_m^2} \mathrm{d}y \, \frac{[q(q_m^2 - y)^3]^{\frac{1}{2}} [y^3 + Ly^2 - My + N]}{(y^2 - Ry + T)^3} . \tag{A.11}$$

The coefficients L, M, N, R, T, V, W and Z are all smoothly varying algebraic functions of α, k and k_0 and the approximate equations (20) and (22) are used in their evaluation. On re-writing

$$y^2 - Ry + T = (y - \alpha_1)(y - \alpha_2) , \tag{A.12}$$

(here α_1, $\alpha_2 > q_m^2$), we find

$$\frac{I_{2,0}}{(4\pi)^2} = \pi \left\{ \frac{(\alpha_1^2 - q_m^2 \alpha_1)^{\frac{1}{2}} - (\alpha_2^2 - q_m^2 \alpha_2)^{\frac{1}{2}}}{(\alpha_1 - \alpha_2)} - 1 \right\} , \tag{A.13}$$

and in the range of interest here, α_1 and α_2 are essentially linear functions of $\eta^2 = k^2/k_0^2$. The integrals for I_1 and $I_{2,1}$ were done numerically. The magnitude of $I_{2,1}$ relative to $I_{2,0}$ varied from 5 % to 15 % in going from near threshold to the highest value of η^2 used ($\eta^2 = 2.00$), so the expansion, equation (A.7) seems justified.

Very near threshold, J_1 and $J_{2,0}$ cancel to order q_m^4 in the case $p(\mathrm{K})$ even, so

$$I_1 - I_{1,0} \xrightarrow[q_m \to 0]{} (4\pi)^2 \frac{40\pi\eta^2}{[\eta^2 + 4(\alpha/k_0)^2]^4} \left(\frac{q_m}{k_0}\right)^6 . \tag{A.14}$$

This is the same order as $I_{2,1}$

$$I_{21} \xrightarrow[q_m \to 0]{} (4\pi)^2 \frac{20\pi}{3} \frac{\eta^2}{[6\eta^2 - 5 + 4(\alpha/k_0)^2]^4} \left(\frac{q_m}{k_0}\right)^6. \tag{A.15}$$

Hence, we get directly for I_-

$$I_- \xrightarrow[q_m \to 0]{} \frac{100\pi}{3} \frac{(4\pi)^2}{[1 + (\alpha/k_0)^2]^4} \left(\frac{q_m}{k_0}\right)^6. \tag{A.16}$$

In the case $p(\mathrm{K})$ odd, very near threshold, I_+ dominates I_-; in I_+, $I_{2,1}$ can be neglected compared to the leading term in $I_1 + I_{2,0}$, so that

$$I_+ \xrightarrow[q_m \to 0]{} \frac{(4\pi)^3}{[1 + 4(\alpha/k_0)^2]^2} \left(\frac{q_m}{k_0}\right)^4. \tag{A.17}$$

RIASSUNTO (*)

Si studia la reazione $K^- + d \to \bar{K}^0 + 2n$ nella speranza che i limiti imposti agli stati finali dei due neutroni dovuti alla statistica aiuteranno a differenziare tra parità pari e dispari dei mesoni K^- e $\bar{K}^0$. Si calcola nell'approssimazione d'impulso la sezione d'urto totale della reazione e la si esprime in termini della corrispondente sezione d'urto dell'idrogeno per le parità sia pari che dispari. Si eseguono i calcoli *a*) con l'ausilio di onde piane antisimmetrizzate per i due neutroni, e *b*) nell'approssimazione di chiusura. Il rapporto della sezione d'urto per parità dispari a quella per parità pari è circa due in ambo le approssimazioni. La sezione d'urto totale per la parità dispari nell'approssimazione in onde piane si trova essere circa 4 mb con un impulso del K^- nel laboratorio di 136 MeV/c.

(*) *Traduzione a cura della Redazione.*

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Formulae for Feynman Graphs of Arbitrary Topology (*).

A. Buccafurri

Scuola di Perfezionamento in Fisica Teorica e Nucleare del C.N.R.N. - Napoli

G. Fano

Istituto di Fisica Teorica dell'Università - Napoli
Scuola di Perfezionamento in Fisica Teorica e Nucleare del C.N.R.N. - Napoli

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Summary. — A compact formula is given, which collects together for any perturbative order, large numbers of topologically equivalent graphs. Problems of enumeration and classification are notably simplified by its use.

We give here a simple formula for the n-th element of the S-matrix, which derives from Caianiello's expansion (1), which expresses in a compact form the contributions of all the Feynman graphs of a given order. Our result can be applied to any two-field theory: electro- or mesodynamics, and many body problems.

Caianiello's formulae were devised and are especially useful for theoretical considerations, because of their complete symmetry. The formulae we deduce here should be advantageous in problems of actual computation, due to their close connection with the standard graphs and to the great simplification in combinatorics which they permit. Our formulae are obtained essentially, by performing on the fermionic part of Caianiello's formula (1) considerations similar—although more involved—to those which, made on the bosonic part,

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(1) E. R. Caianiello: *Nuovo Cimento*, **12**, 561 (1954) formula (30). We will indicate this paper by the letter C.

lead to Caianiello's formula (58). There is, of course, nothing new in principle in our results, except a greater ease of derivation and a synthetic way of writing.

Let us write formula (30) C. which gives the perturbative expansion of a general propagation kernel:

$$(1)\qquad K\begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0} = \sum_{\substack{N=0 \\ (N+P_0 \text{ even})}}^{\infty} \frac{\lambda^N}{N!} \int \sum_{\mu} \gamma^{\mu_1} \dots \gamma^{\mu_N} \, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N \cdot$$

$$\cdot \begin{pmatrix} x_1 x_2 \dots x_{N_0} & \xi_1 \xi_2 \dots \xi_N \\ y_1 y_2 \dots y_{N_0} & \xi_1 \xi_2 \dots \xi_N \end{pmatrix} [t_1 t_2 \dots t_{P_0} \quad \xi_1 \xi_2 \dots \xi_N] \, .$$

We recall that, in electrodynamics:

$$(xy) \quad \text{stands for} \quad \tfrac{1}{2} S_F(xy)$$

$$[xy] \quad \text{stands for} \quad \tfrac{1}{2} \delta\mu_x\mu_y D^F(xy)$$

and $\sum_{\mu}$ means $\sum_{\mu_1\mu_2\dots\mu_N}$.

(The extension to other theories is trivial).

We set:

$$(2)\qquad \begin{cases} (xy)^{(P)} = (x\xi_1)\gamma^{\mu_1}(\xi_1\xi_2)\gamma^{\mu_2} \dots \gamma^{\mu_P}(\xi_P y) & \text{for } p \neq 0 \\ (xy)^{(0)} = (xy) \, , \end{cases}$$

where the integration over the ξ's and the summation over the indices μ are not performed.

We define also:

$$(3)\qquad \sigma_n = \operatorname{Spur} \gamma^{\mu_n}(\xi_n\xi_n)^{(n-1)} = \operatorname{Spur} \gamma^{\mu_n}(\xi_n\xi_1)\gamma^{\mu_1}(\xi_1\xi_2) \dots \gamma^{\mu_{n-1}}(\xi_{n-1}\xi_n) \, .$$

Finally, let us put

$$(4)\qquad \begin{pmatrix} x_1^{(P_1)} & x_2^{(P_2)} & \dots & x_{N_0}^{(P_0)} \\ y_1 & y_2 & \dots & y_{N_0} \end{pmatrix}_s = \begin{vmatrix} (x_1y_1)^{(p_1)} & (x_1y_2)^{(p_1)} & \dots & (x_1y_{N_0})^{(p_1)} \\ (x_2y_1)^{(p_2)} & (x_2y_2)^{(p_2)} & \dots & (x_2y_{N_0})^{(p_2)} \\ \cdot & \cdot & \cdot & \cdot \\ (x_{N_0}y_1)^{(p_{N_0})} & (x_{N_0}y_2)^{(p_{N_0})} & \dots & (x_{N_0}y_{N_0})^{(p_{N_0})} \end{vmatrix}$$

where $s = p_1 + p_2 + \dots p_{N_0}$ and the set of variables ξ is divided into N_0 sub-

sets, as follows:

$\xi_1\xi_2 \dots \xi_{p_1}$ belong to the first subset

$\xi_{p_1+1}\, \xi_{p_1+2} \dots \xi_{p_1+p_2}$ belong to the second subset

. . . .

$\xi_{s-p_{N_0}}\, \xi_{s-p_{N_0}+1} \dots \xi_s$ belong to the N_0-th subset.

We notice that the first subset appears in the first row of the determinant (4), the second subset in the second row, and so on.

We have then the following

THEOREM: The coefficient D_N of λ^N in the expression (1) can be written as

$$D_N = \int \sum_{\mu} [\xi_1\xi_2 \dots \xi_N \;\; t_1 t_2 \dots t_{P_0}] \sum_{s=0}^{N} \frac{(-1)^s}{(N-s)!} \begin{pmatrix} \xi_{s+1}\, \xi_{s+1} \dots \xi_N \\ \xi_{s+1}\, \xi_{s+2} \dots \xi_N \end{pmatrix} \cdot \sum_{p_1 p_2 \dots p_{N_0}} \begin{pmatrix} x_1^{(p_1)}\, x_2^{(p_2)} \dots x_{N_0}^{(p_{N_0})} \\ y_1 \quad y_2 \quad \dots \quad y_{N_0} \end{pmatrix}_s d\xi_1\, d\xi_2 \dots d\xi_N \tag{5}$$

where the $\sum\limits_{p_1p_2\dots p_{N_0}}$ is extended to all non-negative integers such that

$$p_1 + p_2 + \dots p_{N_0} = s\,.$$

The proof of this theorem is in the Appendix.

In this way the determinant of order $N+N_0$ which appears in formula (1) is reduced to a sum of products of two determinants, of order $N-s$ and N_0 respectively. One can further reduce the determinant $\begin{pmatrix} \xi_{s+1}\, \xi_{s+2} \dots \xi_N \\ \xi_{s+1}\, \xi_{s+2} \dots \xi_N \end{pmatrix}$ to a simpler form.

Divide the set of variables $\xi_{s+1} \dots \xi_N$ into l subsets in the following way

1st subset: $\xi_{s+1} \dots \xi_{s+k_1}$,

2nd subset: $\xi_{s+k_1+1} \dots \xi_{s+k_1+2k_2}$,

3rd subset: $\xi_{s+k_1+2k_2+1} \dots \xi_{s+k_1+2k_2+3k_3}$,

. . . .

etc., where k_1, k_2, k_3, must verify the relation

$$k_1 + 2k_2 + 3k_3 + \dots = N - s\,.$$

Now define:

$$\sigma_1^{k_1} = \text{Spur}\, \gamma^{\mu_{s+1}}(\xi_{s+1}\, \xi_{s+1}) \cdot \text{Spur}\, \gamma^{\mu_{s+2}}(\xi_{s+2}\, \xi_{s+2}) \dots \text{Spur}\, \gamma^{\mu_{s+k_1}}(\xi_{s+k_1}\, \xi_{s+k_1}) ,$$

$$\sigma_2^{k_2} = \text{Spur}\, \gamma^{\mu_{s+k_1+1}}(\xi_{s+k_1+1}\, \xi_{s+k_1+2})\gamma^{\mu_{s+k_1+2}}(\xi_{s+k_1+2}\, \xi_{s+k_1+1}) \cdot$$
$$\cdot \text{Spur}\, \gamma^{\mu_{s+k_1+3}}(\xi_{s+k_1+3}\, \xi_{s+k_1+4})\gamma^{\mu_{s+k_1+4}}(\xi_{s+k_1+4}\, \xi_{s+k_1+3}) \dots \cdot$$
$$\cdot \text{Spur}\, \gamma^{\mu_{s+k_1+2k_2-1}}(\xi_{s+k_1+2k_2-1}\, \xi_{s+k_1+2k_2})\gamma^{\mu_{s+k_1+2k_2}}(\xi_{s+k_1+2k_2}\, \xi_{s+k_1+2k_2-1})$$

etc.; obviously, the variables belonging to the n-th subset are those which appear in the expression of $\sigma_n^{k_n}$.

Notice that while the symbol (2) corresponds to a fermion line which is connected to the external points x, y, the σ's correspond to closed fermion lines; thus, in electrodynamics all σ_{2n+1} actually vanish by virtue of Heisemberg's prescription ($n = 0$) and of Furry's theorem ($n \neq 0$).

The reduction of the determinant $\begin{pmatrix} \xi_{s+1}\, \xi_{s+2} \dots \xi_N \\ \xi_{s+1}\, \xi_{s+2} \dots \xi_N \end{pmatrix}$ is made possible by the following identity:

$$(6) \qquad \frac{1}{(N-s)!} \int \sum_{\mu} f(\xi_{s+1}\, \xi_{s+2} \dots \xi_N) \begin{pmatrix} \xi_{s+1}\, \xi_{s+2} \dots \xi_N \\ \xi_{s+1}\, \xi_{s+2} \dots \xi_N \end{pmatrix} \mathrm{d}\xi_{s+1} \dots \mathrm{d}\xi_N =$$
$$= \int \sum_{\mu} f(\xi_{s+1}\, \xi_{s+2} \dots \xi_N) \sum_{k} (-1)^{k_2+k_4+\dots} \frac{\sigma_1^{k_1}}{k_1!} \frac{\sigma_2^{k_2}}{2^{k_2} \cdot k_2!} \frac{\sigma_3^{k_3}}{3^{k_3} k_3!} \mathrm{d}\xi_{s+1} \dots \mathrm{d}\xi^N ,$$

where: $f(\xi_{s+1}\, \xi_{s+2} \dots \xi_N)$ is any symmetric function of $\xi_{s+1} \dots \xi_N$, and the summation $\sum_k$ is extended over all non-negative integers such that

$$N - s = k_1 + 2k_2 + 3k_3 + \dots ;$$

The identity (6) is the obvious extension of things quite well known in the theory of integral equations. A proof of it is given by SCHWINGER (2), whose case differs from ours because:

1) Our σ's contain the variables ξ which are not integrated.

2) Schwinger's proof is for the case $f(\xi_{s+1} \dots \xi_N) = 1$ but a moment's thought shows that the proof holds also in our case.

Substituting eq. (6) into eq. (5) with

$$f = [\xi_1 \xi_2 \dots \xi_N t_1 t_2 \dots t_{\tau_0}]$$

(2) J. SCHWINGER: *Phys. Rev.*, **93**, 615 (1954).

we get:

$$D_N = \int \sum_{\mu} [\xi_1 \xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}] \sum_{s=0}^{N} (-1)^s \sum_{k_1 + 2k_2 + 3k_3 \dots = N-s} (-1)^{k_2 + k_4 + \dots} \cdot$$

$$\cdot \frac{\sigma_1^{k_1}}{k_1!} \frac{\sigma_2^{k_2}}{2^{k_2} \cdot k_2!} \frac{\sigma_3^{k_3}}{3^{k_3} \cdot k_3!} \dots \sum_{p_1 + p_2 + \dots p_{N_0}} \begin{pmatrix} x_1^{(p_1)} & x_2^{(p_2)} & \dots & x_{N_0}^{(p_{N_0})} \\ y_1 & y_2 & \dots & y_{N_0} \end{pmatrix}_s ,$$

or

$$(7) \qquad D_N = \int \sum_{\mu} [\xi_1 \xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}] \sum_{\substack{p_1 + p_2 + \dots p_{N_0} + \\ + k_1 + 2k_2 + 3k_3 + \dots = N}} (-1)^{k_2 + k_4 + \dots + p_1 + p_2 + \dots p_{N_0}} \cdot$$

$$\cdot \frac{\sigma_1^{k_1}}{k_1!} \frac{\sigma_2^{k_2}}{2^{k_2} k_2!} \frac{\sigma_3^{k_3}}{3^{k_3} k_3!} \dots \begin{pmatrix} x_1^{(p_1)} & x_2^{(p_2)} & \dots & x_{N_0}^{(p_{N_0})} \\ y_1 & y_2 & \dots & y_{N_0} \end{pmatrix} .$$

(We notice that $N + P_0$ is even, and that in electrodynamics $N - s$ also is even. Therefore in this case $(-1)^{p_1 + p_2 \dots + p_{N_0}} = (-1)^{P_0}$).

The relation of formula (7) to Feynman graphs is now clear: the factor $\sigma_1^{k_1}$ can be represented by k_1 closed fermion loops with one vertex, the factor $\sigma_2^{k_2}$ can be represented by k_2 closed fermion loops with two vertices, etc.; and the determinant by all the fermion lines connected directly with the external points.

Let us consider a simple case, *i.e.* $N_0 = 1$, and the particular term T with:

$$N = 4 , \qquad p_1 = 2 , \qquad k_2 = 1 ,$$

$$p_2 = p_3 = \dots = 0 , \qquad k_1 = k_3 = \dots = 0 .$$

Apart from a numerical factor this term T is given by (we again refer to electrodynamics):

$$T = \int \sum_{\mu} [\xi_1 \xi_2 \xi_3 \xi_4] (x_1 \xi_1) \gamma^{\mu_1} (\xi_1 \xi_2) \gamma^{\mu_2} (\xi_2 y_1) \cdot \mathrm{Spur}\, \gamma^{\mu_3} (\xi_3 \xi_4) \gamma^{\mu_4} (\xi_4 \xi_3)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4 .$$

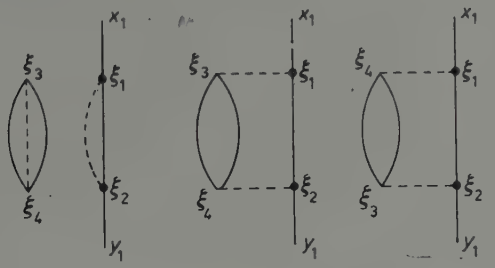
Expanding the hafnian we get (neglecting the factor 1/32):

$$T = \int \sum_{\mu_1 \mu_2 \mu_3 \mu_4} \{ \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} D_F(\xi_1 \xi_2)\, D_F(\xi_3 \xi_4) + \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} D_F(\xi_1 \xi_3)\, D_F(\xi_2 \xi_4) +$$

$$+ \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3} D_F(\xi_1 \xi_4)\, D_F(\xi_2 \xi_3) \} \cdot S_F(x_1 \xi_1) \gamma^{\mu_1} S_F(\xi_1 \xi_2) \gamma^{\mu_2} S_F(\xi_2 y_1) \cdot$$

$$\cdot \mathrm{Spur}\, [\gamma^{\mu_3} S_F(\xi_3 \xi_4) \gamma^{\mu_4} S_F(\xi_4 \xi_3)]\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4 .$$

Therefore the term T corresponds to the three following Feynman graphs:



The advantage of formula (7) is that the numerical coefficient pertaining to each graph is given explicitly. Formula (7) can also be used as a source of approximation techniques: putting

$$\sigma_1 = \sigma_2 = \sigma_3 = \ldots = 0 ,$$

gives only graphs without closed loops; adding the further restriction that the time variables $\xi_{1,0}\, \xi_{2,0} \ldots \xi_{p_1,0}$ are limited to values

$$\xi_{1,0} \geqslant \xi_{2,0} \geqslant \ldots \xi_{p_1,0}$$

(with analogous conditions for the set of variables $\xi_{p_1+1,0} \ldots \xi_{p_1+p_2,0}$, etc.) gives a theory without pairs; etc.

We conclude this note with another formula, deduced from eq. (7), which gives a simplified expression for the element of the S-matrix between two states containing only fermions ($P_0 = 0$). Let us consider electrodynamics; and suppose, for simplicity, to have only electrons in the initial and final states (the generalization to the case in which there are also positrons is trivial).

In order to get the matrix element we must use formula (31) C, with $m = 0$, $h = N_0$, and

$$(8) \qquad \begin{cases} \Phi_I(y_1 y_2 \ldots y_{N_0}) = \dfrac{1}{\sqrt{N_0!}} \begin{pmatrix} u_I^{(1)} u_I^{(2)} \ldots u_I^{(N_0)} \\ y_1 \;\; y_2 \;\; \ldots y_{N_0} \end{pmatrix}, \\ \Phi_F(x_1 x_2 \ldots y_{N_0}) = \dfrac{1}{\sqrt{N_0!}} \begin{pmatrix} u_F^{(1)} u_F^{(2)} \ldots u_F^{(N_0)} \\ x_1 \;\; x_2 \;\; \ldots x_{N_0} \end{pmatrix}. \end{cases}$$

Defining:

$$(9) \qquad \begin{cases} (u_F^{(r)} u_I^{(s)})^{(p_1)} = \overline{u}_F^{(r)}(\xi_1)\gamma^{\mu_1}(\xi_1\xi_2)\gamma^{\mu_2} \ldots \gamma^{\mu_{p_1-1}}(\xi_{p_1-1}\xi_{p_1})\gamma^{\mu_{p_1}} u_I^{(s)}(\xi_{p_1}) , \\ (u_F^{(r)} u_I^{(s)})^{(P_2)} = \overline{u}_F^{(r)}(\xi_{p_1+1})\gamma^{\mu_{p_1+1}}(\xi_{p_1+1}\xi_{p_1+2})\gamma^{\mu_{p_1+2}} \ldots \\ \qquad \ldots \gamma^{\mu_{p_1+p_2-1}}(\xi_{p_1+p_2-1}\xi_{p_1+p_2})\gamma^{\mu_{p_1+p_2}} u_I^{(s)}(\xi_{p_1+p_2}) , \end{cases}$$

etc., it can be easily proved, by using the same methods as in C., that the elements of the S-matrix between the states $|I\rangle$ and $|F\rangle$ specified by the wave functions (8) are given by

$$(10)\qquad M_{FI} = (-1)^{p_{FI}} \frac{1}{N_0!} \sum_{N\,(\text{even})} \lambda^N \int \sum_{\mu} [\xi_1 \xi_2 \dots \xi_N] \sum_{\substack{p_1+p_2+\dots+p_{N_0}+ \\ +2k_2+4k_4+\dots=N}} (-1)^{k_2+k_4+\dots} \frac{\sigma_2^{k_2}}{2^{k_2} k_2!} \frac{\sigma_4^{k_4}}{4^{k_4} k_4!} \dots$$

$$\dots \sum_{l} \begin{pmatrix} u_F^{(p_{l_1})} & u_F^{(p_{l_2})} & \dots & u_F^{(p_{l_{N_0}})} \\ u_I^{(1)} & u_I^{(2)} & \dots & u_I^{(N_0)} \end{pmatrix} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \dots \mathrm{d}\xi_N \,.$$

where $\sum_l$ means the sum over all the permutatùons $l_1 l_2 \dots l_{N_0}$ of the numbers $1, 2, \dots, N_0$, and the values of the elements of the determinant are specified by formulas (9).

Appendix

We give here the proof of a formula, slightly more general than formula (5), which is easier to prove.

By replacing the hafnian $[\xi_1 \xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}]$ with an arbitrary symmetric function $f(\xi_1 \xi_2 \dots \xi_N)$, we prove that, for any M:

$$(\text{A.1})\qquad D'_M = \frac{1}{M!} \int f(\xi_1 \xi_2 \dots \xi_M) \begin{pmatrix} x_1 x_2 \dots x_{N_0} & \xi_1 \xi_2 \dots \xi_M \\ y_1 y_2 \dots y_{N_0} & \xi_1 \xi_2 \dots \xi_M \end{pmatrix} \mathrm{d}\xi_1 \dots \mathrm{d}\xi_M =$$

$$= \int f(\xi_1 \xi_2 \dots \xi_M) \sum_{s=0}^{M} \frac{(-1)^s}{(M-s)!} \begin{pmatrix} \xi_{s+1} \xi_{s+2} \dots \xi_M \\ \xi_{s+1} \xi_{s+2} \dots \xi_M \end{pmatrix} \sum_{p_1+p_2+\dots p_{N_0}=s} \begin{pmatrix} x_1^{(p_1)} & x_2^{(p_2)} & \dots & x_{N_0}^{(p_{N_0})} \\ y_1 & y_2 & \dots & y_{N_0} \end{pmatrix} \mathrm{d}\xi_1 \dots \mathrm{d}\xi_M \,.$$

(We will neglect in the following the γ matrices). We need first the identity (A.2). This we prove by expanding the determinant which appears in the definition (A.1) of D'_M by the elements of the first row; we get (writing N in the place of M):

$$D'_N = \frac{1}{N!} \int \begin{pmatrix} x_1 x_2 \dots x_{N_0} \xi_1 \xi_2 \dots \xi_N \\ y_1 y_2 \dots y_{N_0} \xi_1 \xi_2 \dots \xi_N \end{pmatrix} f(\xi_1 \xi_2 \dots \xi_N) \, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N =$$

$$= \frac{1}{N!} \int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N f(\xi_1 \xi_2 \dots \xi_N) \left\{ \sum_{k=1}^{N_0} (-1)^{k-1} (x_1 y_k) \begin{pmatrix} x_2 \dots & x_{N_0} \xi_1 \xi_2 \dots \xi_N \\ y_1 \dots y_{k-1} y_{k+1} \dots y_{N_0} & \xi_1 \xi_2 \dots \xi_N \end{pmatrix} + \right.$$

$$\left. + \sum_{h=1}^{N} (-1)^{(N_0+h-1)+N_0+h} (x_1 \xi_h) \begin{pmatrix} \xi_h x_2 \dots x_{N_0} \xi_1 \dots \xi_{h-1} \xi_{h+1} \dots \xi_N \\ y_1 y_2 \dots y_{N_0} \xi_1 \dots \xi_{h-1} \xi_{h+1} \dots \xi_N \end{pmatrix} \right\} .$$

Permuting the variables ξ_h and ξ_N and remembering that $f(\xi_1\xi_2 \dots \xi_N)$ is a symmetric function, we get:

$$D'_N = \frac{1}{N!}\int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N\, f(\xi_1\xi_2 \dots \xi_N)\left\{\sum_{k=1}^{N_0}(-1)^{k-1}(x_1y_k)\begin{pmatrix} x_2 \dots & x_{N_0}\xi_1\xi_2 \dots \xi_N \\ y_1 \dots y_{k-1}\, y_{k+1} \dots y_{N_0}\xi_1\xi_2 \dots \xi_N \end{pmatrix} - \right.$$
$$\left. - N\begin{pmatrix} \overset{(1)}{x_1}x_2 \dots x_{N_0}\xi_1\xi_2 \dots \xi_{N-1} \\ y_1y_2 \dots y_{N_0}\xi_1\xi_2 \dots \xi_{N-1} \end{pmatrix}\right\},$$

where clearly:

$$(x_1\, y_l)^{(1)} = (x_1\xi_N)(\xi_N y_l)\,; \qquad (x_1\xi_h)^{(1)} = (x_1\xi_N)(\xi_N\xi_h)\,.$$

But again we can write:

$$\int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N\, f(\xi_1\xi_2 \dots \xi_N)\begin{pmatrix} \overset{(1)}{x_1}x_2 \dots x_{N_0}\xi_1\xi_2 \dots \xi_{N-1} \\ y_1y_2 \dots y_{N_0}\xi_1\xi_2 \dots \xi_{N-1} \end{pmatrix} =$$
$$= \int f(\xi_1\xi_2 \dots \xi_N)\left\{\sum_{k=1}^{N_0}(-1)^{k-1}(x_1y_k)^{(1)}\begin{pmatrix} x_2 \dots & x_{N_0}\xi_4\xi_2 \dots \xi_{N-1} \\ y_1 \dots y_{k-1}\, y_{k+1} \dots y_{N_0}\xi_1\xi_2 \dots \xi_{N-1} \end{pmatrix} + \right.$$
$$\left. + (N-1)\begin{pmatrix} \overset{(2)}{x_1}x_2 \dots x_{N_0}\xi_1\xi_2 \dots \xi_{N-2} \\ y_1y_2 \dots y_{N_0}\xi_1\xi_2 \dots \xi_{N-2} \end{pmatrix}\right\} \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N,$$

where:

$$(x_1y_l)^{(2)} = (x_1\xi_N)(\xi_N\xi_{N-1})(\xi_{N-1}y_l)\,, \quad \text{etc.}$$

Iteration of this procedure leads to the required identity:

$$\text{(A.2)} \qquad D'_N = \int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N\, f(\xi_1\xi_2 \dots \xi_N)\sum_{p=0}^{N}(-1)^p\,\frac{1}{(N-p)!}\sum_{k=1}^{N_0}(-1)^{k-1}(x_1y_k)^{(p)}\cdot$$
$$\cdot\begin{pmatrix} x_2x_3 \dots & x_{N_0}\xi_1\xi_2 \dots \xi_{N-p} \\ y_1 \dots y_{k-1}\, y_{k+1} \dots y_{N_0}\xi_1\xi_2 \dots \xi_{N-p} \end{pmatrix}.$$

The proof is by induction. Suppose that the relation (A.1) holds where in the place of N_0 we write $N_0 - 1$. Then substitution of eq. (A.1) with $M = N - p$ into the identity (A.2) gives

$$D'_N = \int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N\, f(\xi_1\xi_2 \dots \xi_N)\sum_{p=0}^{N}(-1)^p\,\frac{1}{(N-p)!}\sum_{k=1}^{N_0}(-1)^{k-1}(x_1y\ )^{(p)}(N-p)!\cdot$$
$$\cdot\sum_{r=0}^{N-p}\frac{(-1)^r}{(N-r-p)!}\begin{pmatrix} \xi_{r+1}\,\xi_{r+2} \dots \xi_{N-p} \\ \xi_{r+1}\,\xi_{r+2} \dots \xi_{N-p} \end{pmatrix}\sum_{p_1+p_2+\dots p_{N_0-1}=r}\begin{pmatrix} \overset{(p_1)}{x_1}\ \overset{(p_2)}{x_2} \dots & \overset{(p_{N_0-1})}{x_{N_0}} \\ y_1\ y_2 \dots y_{k-1}\, y_{k+1} \dots y_{N_0} \end{pmatrix}.$$

We can replace $\xi_{r+1} \dots \xi_{N-p}$ by $\xi_{r+p+1} \dots \xi_N$ in the first determinant, and the dummy variables of $(x_1 y_k)^{(p)}$ by $\xi_{r+1} \dots \xi_{r+p}$. Then putting $r+p=s$, we get:

$$D'_N = \int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N \, f(\xi_1 \xi_2 \dots \xi_N) \sum_{k=1}^{N_0} (-1)^{k-1} \sum_{s=0}^{N} \sum_{p=0}^{s} \frac{(-1)^s}{(N-s)!} \cdot$$

$$\cdot \begin{pmatrix} \xi_{s+1} \xi_{s+2} \dots \xi_N \\ \xi_{s+1} \xi_{s+2} \dots \xi_N \end{pmatrix} \cdot (x_1 y_k^{(p)}) \sum_{p_1+p_2+\dots p_{N_0}-1=s-p} \begin{pmatrix} {}^{(p_1)} {}^{(p_2)} & & {}^{(p_{N_0}-1)} \\ x_2 \; x_3 \dots & & x_{N_0} \\ y_1 \; y_2 \dots y_{k-1} \, y_{k+1} \dots & & y_{N_0} \end{pmatrix}.$$

Taking first the sum with respect to k we can reproduce the determinant

$$\begin{pmatrix} {}^{(p_1)} {}^{(p_2)} & & {}^{(p_{N_0})} \\ x_1 \; x_2 \dots & & x_{N_0} \\ y_1 \; y_2 \dots & & y_{N_0} \end{pmatrix};$$

we get

$$D'_N = \int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_N \, f(\xi_1 \xi_2 \dots \xi_N) \sum_{s=0}^{N} \frac{(-1)^s}{(N-s)!} \begin{pmatrix} \xi_{s+1} \xi_{s+2} \dots \xi_N \\ \xi_{s+1} \xi_{s+2} \dots \xi_N \end{pmatrix} \cdot$$

$$\cdot \sum_{p_1+p_1+\dots p_{N_0}-1=s} \begin{pmatrix} {}^{(p)} {}^{(p_1)} {}^{(p_2)} & & {}^{(p_{N_0}-1)} \\ x_1 \; x_2 \; x_3 \dots & & x_{N_0} \\ y_1 \; y_2 & \dots & y_{N_0} \end{pmatrix}.$$

which coincides with eq. (A.1).

For $N_0 = 0$ the relation (A.1) becomes an identity (because $p_1 = p_2 = \dots = p_{N_0} = s = 0$). **Q.E.D.**

RIASSUNTO (*)

Si dà una formula complessa che comprende per ogni ordine di perturbazione un gran numero di grafici topologicamente equivalenti. I problemi di enumerazione e classificazione ne risultano notevolmente semplificati.

(*) *Traduzione a cura della Redazione.*

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Regularization and Renormalization.

I – General Part (*).

E. R. Caianiello

Istituto di Fisica dell'Università - Napoli
Scuola di Perfezionamento in Fisica Teorica e Nucleare del C.N.R.N. - Napoli

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Summary. — A general formalism is presented which permits to investigate the question, under which conditions does a regularization procedure become a renormalization. This is the case whenever the procedure satisfies conditions of two sorts: a first set of conditions is of very general nature and stems only from combinatorial and analytical requirements; a second set restricts both field theories and procedures alike. This work gives an exhaustive treatment of the first set of conditions and prepares the ground for the study of the second set, which will soon be published. The main conclusion will be that the problem of the renormalization of ultraviolet divergencies can be completely bypassed by an appropriate redefinition of the concept of integral, such as is required when hyperbolic equations are considered. The treatment is mathematically rigorous and wholly independent of perturbative methods; it puts the theory on a firm footing for subsequent quantitative studies.

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1. – Introduction.

1·1. – An exhaustive treatment of the formal theory of quantized fields has been given in a series of works [1], the main purpose of which was to show:

a) that any perturbative expansion can be written in compact form if two—and only two—new algorithms—the pfaffian for Fermi fields, the hafnian for Bose fields—are introduced;

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(1) E. R. Caianiello: *Nuovo Cimento*, **10**, 1634 (1953); **11**, 492 (1954); **12**, 561 (1954); **2**, 186 (1955); **3**, 223 (1956); **5**, 739 (1957); **8**, 170 (1958) (whith A. Buccafurri).

b) that the study of a field theory is conveniently reduced to that of infinite sets of coupled hyperbolic equations—named by us « branching equations », because of their characteristic structure—which are satisfied by the propagation kernels (simply « kernels » in our terminology); the perturbative expansions of the kernels are formal solutions of these equations.

This approach reduces the order of difficulty of the problem, because it suffices then to investigate properties of functions—or distributions—rather than of field operators.

The branching equations can be taken, in turn, as the axiomatic formulation of the theory. They may be regarded as the natural generalization—which includes all possible cases—of the Fredholm equation (to which they reduce in the case of a Fermi field interacting with an external Bose field: in this case pfaffians reduce to the determinants of the Fredholm theory). This is the point of view we take here; although it is formally equivalent to the standard procedure of deducing all equations from a Lagrangian, under some assumptions on the existence of a vacuum state, of canonical commutation relations, of relativistic covariance, no such hypothesis need be made explicitly if one starts from a set of branching equations as the first thing: it may well happen that solutions to the latter exist which do not satisfy the standard postulates just mentioned—or, for all we know at present, the opposite may be true. These questions will become of great interest when it will be possible to deal with them quantitatively.

The present work is concerned with the next major problem of the theory of quantized fields, the renormalization of ultraviolet divergencies. The history of this subject is too well known to bear repetition; we just remark that the beautiful work of FEYNMAN [2], SCHWINGER [3], DYSON [4], and many others, which has taught how to circumvent in practical computations the troubles caused by the appearance of infinite quantities of this nature, by removing them consistently into parameters to which finite values are imposed *a posteriori* as the observed experimental values, has not solved—nor was it intended to solve—the fundamental question, whether such infinities were due to inadequacy of the mathematical methods used, or were rather intrinsic in the physics itself. The question is indeed fundamental, because in the latter case no existing field theory, no matter how renormalizable, could be considered as self-consistent: if « bare » masses and charges are actually infinite, then there must be a theory behind the theory, which completes it, either with the adjunction of more fields or with the attribution of an inner structure to the particles, so that only the total « final » theory is written, satisfactorily, in

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[2] R. P. FEYMAN: *Phys. Rev.*, **76**, 769 (1949).
[3] J. SCHWINGER: *Phys. Rev.*, **74**, 1439 (1948); **75**, 651 (1949).
[4] F. J. DYSON: *Phys. Rev.*, **75**, 486 (1949); **75**, 1736 (1949).

terms of finite parameters. All attempts in this direction have, however, failed. It is quite reasonable, on the other hand, to expect that interactions, say, of electrons with heavier particles, should amount, in a theory like electrodynamics, only to minor numerical corrections to the actual theory, which considers only electrons and photons.

Of great interest is, in this connection, KÄLLÉN's result (5), obtained by using the customary Green functions and the notion of Riemann integral, that at least one of the renormalization constants is infinite. We shall agree entirely with his finding, that use of the standard concept of integral requires infinite counter-terms in the theory, independently of perturbative treatments. We shall show, however, that if the very concept of integral is modified *as is required by the rigorous theory of hyperbolic equations*—by some extension of the Cauchy principal value integral, or by straightforward use of Hadamard's « partie finie » (6)—then all ultraviolet divergencies disappear consistently; and that if the theory satisfies some conditions—which are at worst Dyson's conditions for renormalizability—then all the infinite terms thus avoided are identical with those that the standard counter-terms would have introduced.

1'2. – Some of the results reported here have been already anticipated without proofs elsewhere (7). We think it convenient, though, to present here a systematic and self-contained treatment of the subject, which may spare the reader the trouble of referring to other works for notation and basic concepts; some subsections of Section **2** are accordingly dedicated to a very brief summary of results which are relevant for our purposes, a full account of which is found in ref. (1). Section **2** contains also additional algebraic theorems, and their applications to kernels, which are needed in this work.

In Section **3** we examine some typical problems which arise when multiple integrals are regularized; some combinatorial devices, to be applied later on to kernels, are analyzed in detail on a simple example. This prepares the ground for Section **4**, where we continue with the central question, under which conditions does a regularization act as a renormalization. By regularization we mean here any procedure whatsoever which, regardless of physical motivations, changes an otherwise divergent integral into a convergent one. Any such procedure amounts to considering the original divergent (Riemann or Lebesgue) integral I as the sum of two parts: one, I_R, which is defined through the regularization rule itself, be this what it may, so that it stays

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(5) G. KÄLLÉN: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **27**, no. 12 (1953).

(6) J. HADAMARD: *Le Problème de Cauchy* (Paris 1932).

(7) E. R. CAIANIELLO: *Suppl. Nuovo Cimento*, **9**, 569 (1958). See also *Max-Planck Festschrift*, 1958.

finite; the other, I_D, which is infinite and can therefore only be defined formally as $I_D = I - I_R$, or as an (infinite) limit if a parameter appears in the regularization rule.

Regularization becomes a renormalization if it can be proved, in addition, that all terms I_D which are dropped by its adoption combine together, formally, to give contributions to the parameters—masses and charges—in terms of which the theory was written originally. For this to happen, one expects therefore conditions of two distinct sorts: a first set, which depends only on symmetry, iterability, etc., and is therefore of a general and purely mathematical character; a second set, which depends also upon the physical theory under consideration. In Section **4** we study the first type of conditions and find that very general qualitative requirements on the regularization procedure suffice to produce the typical decomposition of kernels, which will provide the basis for the subsequent analysis. Section **4** is concluded with the discussion of another example, in which regularization amounts simply to taking principal value integrals, which shows in a perspicuous way how the transition from ordinary to regularized integrals serves to render meaningful an otherwise inconsistent theory and gives rise to phenomena which were supposed before to be typical of standard renormalization.

The case of actual field theories involves several complications; it will be dealt with exhaustively in the following article which will complete this part of our investigation.

2. – Summary of previous results. Algebraic theorems.

In this Section we give, for convenience of the reader, a brief account of results obtained in the past [1], which are relevant for our work. We report also some new algebraic theorems on the expansion of pfaffians, hafnians, etc., which will facilitate our analysis of renormalization.

2'1. – Notation is a matter of prime importance. Our conventions are intended to achieve maximum economy and generality. At a stage or another, all field-theoretical calculations require that one takes expectation values of products of free fields operators between arbitrary states; to determine these, it suffices to know the vacuum expectation values. We are interested only in time-ordered products of field operators, and report therefore only the formulae which pertain to this case; any other type of product gives rise, though, to similar results—a fact which here can only be mentioned. We refer to electrodynamics as a special example; the generality of the formalism should, however, be evident.

We propose to study the elements, between arbitrary initial and final states, of the matrix:

$$(1) \qquad U(t_F, t_I) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int_{\Omega} \mathrm{d}^4\xi_1 \dots \int_{\Omega} \mathrm{d}^4\xi_N T\Big(\prod_{i=1}^{N} \overline{\psi}_{\alpha_i}(\xi_i)\gamma^{\mu_i}_{\alpha_i\beta_i}\psi_{\beta_i}(\xi_i)A_{\mu_i}(\xi_i)\Big) ,$$

where: $\Omega \equiv \{V,\ T = t_F - t_I)\}$; $\lambda =$ el. charge; $\hbar = c = 1$; $U \equiv S$ for $t_{F,I} = \pm\infty$; the fields A and ψ commute (interaction representation).

We have then to consider the vacuum expectation values:

$$(2) \qquad F_{2N} = \langle 0 \,|\, T(\overline{\psi}_{\alpha_1}(x_1)\psi_{\beta_1}(x_1) \dots \overline{\psi}_{\alpha_N}(x_N)\psi_{\beta_N}(x_N)) \,|\, 0\rangle = (1\ 2 \dots 2N)$$

and

$$(3) \qquad B_N = \langle 0 \,|\, T(A_{\mu_1}(x_1)A_{\mu_2}(x_2) \dots A_{\mu_N}(x_N)) \,|\, 0\rangle = \begin{cases} 0 \colon N \text{ odd} \\ [1\ 2 \dots 2M] \colon N = 2M . \end{cases}$$

The symbols $(1\ 2 \dots 2N)$ and $[1\ 2 \dots 2M]$ which occur in (2) and (3) introduce two algorithms:

a) Pfaffians. These suffice to express any type of expectation value of products of free anticommuting fields, of which (2) is only a special instance. Their general definition is (we use round brackets for $\pm$ expansion rules, square brackets for only $+$ expansion rules):

$$(4) \qquad (1\ 2 \dots 2n) = \sum_{h=2}^{2n} (-1)^h (1h)(2 \dots h-1,\ h+1 \dots 2n) =$$
$$= \sum{}' (-1)^p (i_1 i_2)(i_3 i_4) \dots (i_{2n-1} i_{2n}) ,$$

where $\sum'$ means sum over all permutations $i_1 i_2 \dots i_{2n}$ of $1\ 2 \dots 2n$ (of parity p) such that $i_1 < i_3 < \dots\dots < i_{2n-1}$ and $i_1 < i_2$, $i_3 < i_4$, ..., $i_{2-n1} < i_{2n}$). They are conveniently represented by the symbol

$$(5) \qquad (1\ 2 \dots 2n) = \begin{vmatrix} (1\ 2) & (1\ 3) & \dots & (1, 2n) \\ & (2\ 3) & \dots & (2, 2n) \\ & & \ddots & \vdots \\ & & & (2n-1,\ 2n) \end{vmatrix} .$$

b) Hafnians. These suffice, likewise, to express any type of expectation values of products of free commuting fields:

$$(6) \qquad [1\ 2 \dots 2n] = \sum_{h=2}^{2n} [1h]\ [2 \dots h-1,\ h+1 \dots 2n] \text{ etc.},$$

where it suffices to replace all $\pm$ signs in (4) with $+$.

We further use for determinants the notation (Sylvester, Cayley):

$$(7) \qquad \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = \begin{vmatrix} (11) & (12) & \dots & (1n) \\ (21) & (22) & \dots & (2n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (n1) & (n2) & \dots & (nn) \end{vmatrix}.$$

It can then be shown that the particular pfaffian (2) reduces to:

$$(2') \qquad F'_{2n} = \begin{pmatrix} 1 & 2 & \dots & N \\ 1 & 2 & \dots & N \end{pmatrix},$$

where

$$(8) \qquad (hk) = \frac{1}{2} \int_{\beta_h \alpha_k}^{F} (x_h - y_k) ; \qquad (\gamma \partial_x + m_f)(xy) = i\delta(x-y) .$$

(3) is given by (6) where

$$(9) \qquad [hk] = \tfrac{1}{2} \delta_{\mu_h \mu_k} D^F(x_h - x_k); \qquad (\Box_x - m_b^2)[xy] = i\,\delta(x-y)$$

(or $= \frac{1}{2} D^F_{\mu_h \mu_k}(x_h - x_k)$ in a general gauge).

Finally, we use for permanents (determinants expanded with a + sign rule) the symbol

$$(10) \qquad \begin{bmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{bmatrix}.$$

The four algorithms just described cover all cases that can occur in any field theory; pfaffians and hafnians are primitive, determinants and permanents occur as degenerate cases.

It was shown [1] that the most general element of (1) can be calculated easily from the kernels $K_{N_0 P_0}$ (vacuum expectation values of time-ordered products of Heisenberg fields, multiplied by K_{00}), the perturbative expansion of which is, in cases like electrodynamics:

$$(11) \qquad K_{N_0 P_0} = K \begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0} \Big) =$$

$$= \sum_{\Lambda(P_0)} \frac{\lambda^N}{N!} \int \mathrm{d}\xi_1 \dots \int \mathrm{d}\xi_N \sum \gamma^1 \dots \gamma^N \begin{pmatrix} x_1 \dots x_{N_0} & \xi_1 \dots \xi_N \\ y_1 \dots y_{N_0} & \xi_1 \dots \xi_N \end{pmatrix} [t_1 \dots t_{P_0}\ \xi_1 \dots \xi_N] ,$$

where: $\gamma^h \equiv \gamma^{\mu_h}_{\alpha_h \beta_h}$; $\sum_{N(P_0)}$ is sum over all N with same parity as P_0 (no. of external boson lines).

2'2. – The fundamental expansion theorems for pfaffians and hafnians, which are the counterpart of the theorems relative to the expansion of determinants by minors of given order, are:

$$(12)\qquad (1\,2\,\ldots\,2n) = \sum_{r=0}^{[m/2]} (-1)^{\binom{m-2r}{2}} \sum_{C_r} \sum_{C_t} (-1)^{p(h,k)} (h'_1, \ldots, h'_{2r}) \begin{pmatrix} h''_1 \ldots h''_s \\ k''_1 \ldots k''_s \end{pmatrix} (k'_1 \ldots k'_{2t}),$$

and

$$(13)\qquad (1\,2\,\ldots\,2n) = \sum_{r=0}^{[m/2]} \sum_{C_r} \sum_{C_t} [h'_1 \ldots h'_{2r}] \begin{bmatrix} h''_1 \ldots h''_t \\ k''_1 \ldots k''_s \end{bmatrix} [k'_1 \ldots k'_{2t}],$$

where:

m is the number $\leqslant 2n$ of the lines in the pfaffian or hafnian (line h contains all elements with the index h, whether in the first or in the second place) which are selected for the expansion;

$h'_1 < h'_2 < \ldots < h'_{2r}$; $h''_1 < h''_2 < \ldots < h''_s$ is any combination of the indices which characterize those m lines; same for $k'_1 < \ldots < k'_{2t}$; $k''_1 < \ldots < k''_s$; $\sum_{C_r}$ and $\sum_{C_t}$ denote sums over all such combinations;

$p(h, k)$ is the parity of the permutation $h'_1 \ldots h'_{2r}\, h''_1 \ldots h''_s\, k''_1 \ldots k''_s\, k'_1 \ldots k'_{2t}$ with respect to $1\,2\,\ldots\,2m$;

$[m/2]$ is the standard symbol for the maximum integer contained in $m/2$; $m = 2r+s$; $s+2t = 2n-m$.

It is important for our purposes to introduce an additional notational convention. Define first the symbols:

$$(14)\qquad \begin{cases} (\mathring{h}\mathring{k}) = 0\,, & (\hat{h}k) = (h\mathring{k}) = (hk)\,; \\ [\mathring{h}\mathring{k}] = 0\,, & [\hat{h}k] = [h\mathring{k}] = [hk]\,; \end{cases}$$

then, for instance: $\begin{pmatrix} \hat{1}\ \hat{2}\ 3\ 4 \\ \hat{1}\ \hat{2}\ 3\ 4 \end{pmatrix}$ denotes the determinant $\begin{pmatrix} 1\ 2\ 3\ 4 \\ 1\ 2\ 3\ 4 \end{pmatrix}$ after all its elements (11), (12), (21), (22) are replaced by zero; in general, this is simply shorthand to give compact form to expressions which are defined by their formal expansion rules and, in addition, (14).

We have then the remarkable expansions (defined here, for simplicity, only in the case that the lines selected are the *first* m):

$$(15) \qquad (1\ 2\ \dots\ 2n) = \sum_{r=0}^{[m/2]} \sum_{\sigma_r} (-1)^{p(h)} (h'_1 \dots h'_{2r})(\overset{\circ}{h}{}''_1 \dots \overset{\circ}{h}{}''_s\ m+1 \dots 2n)\,,$$

$$(16) \qquad [1\ 2\ \dots\ 2n] = \sum_{r=0}^{[m/2]} \sum_{\sigma_r} [h'_1 \dots h'_{2r}][\overset{\circ}{h}{}''_1 \dots \overset{\circ}{h}{}''_s\ m+1 \dots 2n]\,,$$

with the same notation introduced above, and $p(h)$ parity of $h'_1 \dots h'_{2r}\ h''_1 \dots h''_s$ with respect to $1\ 2 \dots m$.

The proof is trivial: (16) follows directly from (13) applied to an expansion by the zeroed lines of

$$[\overset{\circ}{h}{}''_1 \dots \overset{\circ}{h}{}''_s\ m+1 \dots 2n] = \sum_{\sigma_t} \begin{bmatrix} h''_1 \dots h''_s \\ k''_1 \dots k''_s \end{bmatrix} [k'_1 \dots k'_{2t}]\,,$$

(15) from a similar expansion and the remark that, here, $p(h, k) = p(h) + p(k)$ and $s = m - 2r$.

We have, finally, similar expansions for determinants and permanents:

$$(17) \qquad \begin{pmatrix} 1 \dots m\ m+1 \dots n \\ 1 \dots m\ m+1 \dots n \end{pmatrix} = \sum_{r=0}^{m} \sum_{\sigma_r^R} \sum_{\sigma_r^C} (-1)^{p(h,k)} \begin{pmatrix} h'_1 \dots h'_r \\ k'_1 \dots k'_r \end{pmatrix} \begin{pmatrix} \overset{\circ}{h}{}''_1 \dots \overset{\circ}{h}{}''_s\ m+1 \dots n \\ \overset{\circ}{k}{}''_1 \dots \overset{\circ}{k}{}''_s\ m+1 \dots n \end{pmatrix},$$

and

$$(18) \qquad \begin{bmatrix} 1 \dots m\ \ m+1 \dots n \\ 1 \dots m\ \ m+1 \dots n \end{bmatrix} = \sum_{r=0}^{m} \sum_{\sigma_{r_i}^R} \sum_{\sigma_r^C} \begin{bmatrix} h'_1 \dots h'_r \\ k'_1 \dots k'_r \end{bmatrix} \begin{bmatrix} \overset{\circ}{h}{}''_1 \dots \overset{\circ}{h}{}''_s\ m+1 \dots n \\ \overset{\circ}{k}{}''_1 \dots \overset{\circ}{k}{}''_s\ m+1 \dots n \end{bmatrix},$$

where $h'_1 < h'_2 < \dots < h'_r$; $h''_1 < \dots < h''_s$ is any combination Π_h of the row indices $1\ 2 \dots m$; $k'_1 < \dots < k'_2$; $h''_1 < \dots < k''_s$ any combination Π_k of the column indices $1\ 2 \dots m$; $p(h, k)$ is the parity of Π_h with respect to Π_k; $\sum_{\sigma_r^R}$ and $\sum_{\sigma_r^C}$ sum over all such permutations.

The proofs are again quite simple with this formalism; we need not report them here, since (17) is, in an improved notation, the classical theorem of Arnaldi, and (18) is an obvious adaptation of it to permanents.

2·3. – The theorems (15) to (18) are of importance because of their applications to kernels. As is always the case, all formal results deduced from the perturbative expansions can be proved to hold in general, directly from the branching equations. Obvious manipulations give, in the case of electrodynamics, starting from the fundamental expansion formula for kernels (11)

and applying (16) and (17) to hafnians and determinants:

$$(19)\qquad K\left(\begin{matrix}\xi_1 \dots \xi_m & x_1 \dots x_n \\ \eta_1 \dots \eta_m & y_1 \dots y_n\end{matrix}\middle|\, \tau_1 \dots \overset{h}{\tau}_q \quad t_1 \dots t_p\right) =$$

$$= \sum_{r=0}^{m} \sum_{\varrho=0}^{[q/2]} \sum_{c_r^R} \sum_{c_r^C} \sum_{c_\varrho} (-1)^{p(h,k)} \begin{pmatrix} \xi_{h_1'} \dots \xi_{h_r'} \\ \eta_{k_1'} \dots \eta_{k_r'} \end{pmatrix} [\tau_{l_1'} \dots \tau_{l_{2\varrho}'}] \cdot$$

$$\cdot K\left(\begin{matrix} \overset{\circ}{\xi}_{h_1''} \dots \overset{\circ}{\xi}_{h_s''} & x_1 \dots x_n \\ \overset{\circ}{\eta}_{k_1''} \dots \overset{\circ}{\eta}_{k_s''} & y_1 \dots y_n \end{matrix}\middle|\, \overset{\circ}{\tau}_{l_1''} \dots \overset{\circ}{\tau}_{l_\sigma''} \quad t_1 \dots t_p\right),$$

where: $h_1' < \dots < h_r'$; $h_1'' < \dots < h_s''$ is a combination of $1 \dots m$, etc., $r+s=m$; $2\varrho+\sigma=q$, and the kernels with symbols $\overset{\circ}{\xi}_{h''}$, etc., are defined by (11), or, alternatively, by the equations reported in the next section and (14). Physically, these kernels describe the same processes as the originary kernels, except that they do not contain those processes which are described by disjoint graphs having free fermion lines terminating in points $\overset{\circ}{\xi}_{h''}$, $\overset{\circ}{\eta}_{k''}$, and free boson lines terminating in points $\overset{\circ}{\tau}_{l''}$.

2·4. – In the formal theory the branching equations among kernels can be written either as recurring hyperbolic differential equations with specific boundary conditions (in our case, causality requirements), or, equivalently, as recurring integral equations, the solutions of which (if any) satisfy both the differential equations and the boundary conditions. It is, however, in the indiscriminate transition from the first to the second way of formulating the problem that most of the troubles due to the so-called ultraviolet divergencies are originated; as is well known, indeed, this formal transition is entirely meaningless already in the case of ordinary hyperbolic equations.

The case of a quantized electron field in an external E.M. field—in which the first traces of the kernel of the corresponding Fredholm equation are infinite, so that one must add special prescriptions, which were called also renormalization before their trivial origin was made evident (8)—is also instructive. The correct transition from differential to integral equations obtains only if special cautions are taken, such as using a re-defined concept of integral, as first introduced by HADAMARD (6).

One has then the choice, to start directly from the differential formulation of the problem and devise correct methods for its solutions, or to reconsider the integral branching equations so as to remove all ambiguities and troubles

(8) P. T. MATTHEWS and A. SALAM: *Phys. Rev.*, **90**, 690 (1953).

from their definition. We follow the second approach, which is more convenient and, of course, entirely equivalent to the first one.

We record here, for reference, some equations among kernels which are of special interest to us; then, in the next section, we state a few lemmas on hafnians and determinants which are needed later. This will complete the necessary preliminaries for our discussion.

Branching equations among kernels:

$$(20)\qquad K\begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0} = \sum_{h=2}^{P_0} [t_1 t_h] K \begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_2 \dots t_{h-1}\, t_{h+1} \dots t_{P_0} + \\ + \lambda \int \mathrm{d}\xi \sum \gamma^{\xi} [t_1\, \xi] K \begin{pmatrix} x_1 \dots x_{N_0} \xi \\ y_1 \dots y_{N_0} \xi \end{pmatrix} t_2 \dots t_{P_0},$$

$$(21)\qquad K_{N_0 P_0} = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) K \begin{pmatrix} x_2 \dots x_{N_0} \\ y_1 \dots y_{h-1}\ y_{h+1} \dots y_{N_0} \end{pmatrix} t_1, \dots t_{P_0} - \\ - \lambda \int \mathrm{d}\xi \sum \gamma^{\xi} (x_1\, \xi) K \begin{pmatrix} \xi\ x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} \xi_1 t_1 \dots t_{P_0},$$

(and the associate equation),

$$(22)\qquad K\begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) K \begin{pmatrix} x_2 \dots x_{N_0} \\ y_1 \dots y_{h-1}\ y_h \dots y_{N_0} \end{pmatrix} - \\ - \lambda^2 \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \sum \gamma^{\xi_1} \gamma^{\xi_2} (x_1 \xi_1) [\xi_1 \xi_2] K \begin{pmatrix} \xi_1 x_2 \dots x_{N_0} \xi_2 \\ y_1 y_2 \dots y_{N_0} \xi_2 \end{pmatrix}.$$

Branching equations involving derivatives of kernels:

$$(23)\qquad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int \mathrm{d}\xi \sum \gamma^{\xi} K \begin{pmatrix} x_1 \dots x_{N_0} \xi \\ y_1 \dots y_{N_0} \xi \end{pmatrix} \xi t_1 \dots t_{P_0},$$

$$(24)\qquad \frac{\partial K_{N_0 P_0}}{\partial m_f} = -i \int \mathrm{d}\xi\, K \begin{pmatrix} \xi x_1 \dots x_{N_0} \\ \xi y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0},$$

$$(25)\qquad \frac{\partial K_{N_0 P_0}}{\partial (m_b^2)} = -\frac{i}{2} \int \mathrm{d}\xi\, K \begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} \overset{\circ}{\xi}\overset{\circ}{\xi} t_1 \dots t_{P_0}.$$

(24) and (25) are the counterpart of Ward's identities.

2·5. – The following properties of hafnians and determinants will be useful in the study of electrodynamics.

a) Lemmas on hafnians. Let $[hk]=[kh]$ (as is the case with time-ordered expansions); then, if $\sum_{C_\varrho}$ is the sum over all the C_ϱ combinations $l'_1<\ldots<l'_{2\varrho}$; $l''_1<\ldots<l''_\sigma$ of the indices $\alpha_1\alpha_2\ldots\alpha_\mu$ $(\mu=2\varrho+\sigma)$, and $\sum_{C_\tau}$ is the sum over all C_τ combinations $m''_1<\ldots<m''_\sigma$; $m'_1<\ldots<m'_{2\tau}$ of the indices $\beta_1\beta_2\ldots\beta_\nu$ $(\nu=2\tau+\sigma)$:

$$(26)\qquad \sum_{C_\varrho}[l'_1l'_2\ldots l'_{2\varrho}][\overset{\circ}{l}''_1\ldots\overset{\circ}{l}''_\sigma\ \beta_1\beta_2\ldots\beta_\nu]=\sum_{C_\tau}[\alpha_1\alpha_2\ldots\alpha_\mu\ \overset{\circ}{m}''_1\ldots\overset{\circ}{m}''_\sigma][m'_1\ldots m'_{2\tau}]\,.$$

The proof of (26) is an immediate consequence of (16).

Also $(\sigma=2)$:

$$(27)\qquad \sum_{C_\tau}[m''_1m''_2][m'_1\ldots m'_{2\tau}]=\frac{\nu}{2}[\beta_1\beta_2\ldots\beta_\nu]$$

and

$$(28)\qquad \sum_{C_\tau}\frac{\partial[m''_1m''_2]}{\partial(m_b^2)}[m'_1\ldots m'_{2\tau}]=\frac{\partial}{\partial(m_b^2)}[\beta_1\beta_2\ldots\beta_\nu].$$

(28) follows from $\sum_{C_\tau}=\sum_{m''_1<m''_2}$; (27) from this and $\sum_{m''_1<m''_2}=\frac{1}{2}\sum_{m''_1\neq m''_2}$.

b) Lemmas on determinants. These are consequences of Arnaldi's theorem (17); $p(i,j)$ has meaning similar to that of $p(h,k)$ stated there.

$\sum_{C_r}$ denotes summation over all combinations $h'_1<\ldots<h'_r$; $h''_1<\ldots<h''_s$ of $\alpha_1\ldots\alpha_m$ $(m=r+s)$ and $k'_1<\ldots<k'_r$; $k''_1<\ldots<k''_s$ of $\overline{\alpha}_1\ldots\overline{\alpha}_m$; $\sum_{C_t}$ over all combinations $i''_1<i''_2<\ldots<i''_s$, $i'_1<\ldots<i'_t$ of $\beta_1\ldots\beta_n$ $(n=s+t)$ and $j''_1<\ldots<j''_s$, $j'_1<\ldots<j'_t$ of $\overline{\beta}_1\ldots\overline{\beta}_n$.

Then

$$(29)\qquad \sum_{C_r}(-1)^{p(h,k)}\begin{pmatrix}h'_1\ldots h'_r\\ k'_1\ldots k'_r\end{pmatrix}\begin{pmatrix}\overset{\circ}{h}''_1\ldots\overset{\circ}{h}''_s & \beta_1\ldots\beta_n\\ \overset{\circ}{k}''_1\ldots\overset{\circ}{k}_s & \overline{\beta}_1\ldots\overline{\beta}_n\end{pmatrix}=$$

$$=\sum_{C_t}(-1)^{p(i,j)}\begin{pmatrix}\alpha_1\ldots\alpha_m & \overset{\circ}{i}''_1\ldots\overset{\circ}{i}''_s\\ \overline{\alpha}''_1\ldots\overline{\alpha}_m & \overset{\circ}{j}''_1\ldots\overset{\circ}{j}''_s\end{pmatrix}\begin{pmatrix}i'_1\ldots i'_t\\ j'_1\ldots j'_t\end{pmatrix}.$$

Also $(s=1)$:

$$(30)\qquad \sum_{C_t}(-1)^{p(i,j)}(i''j'')\begin{pmatrix}i'_1\ldots i'_t\\ j'_1\ldots j'_t\end{pmatrix}=n\begin{pmatrix}\beta_1\ldots\beta_n\\ \overline{\beta}_1\ldots\overline{\beta}_n\end{pmatrix},$$

and

$$\sum_{C_t} (-1)^{p(i,j)} \frac{\partial(i''j'')}{\partial m_f} \begin{pmatrix} i'_1 \dots i'_t \\ j'_1 \dots j'_t \end{pmatrix} = \frac{\partial}{\partial m_f} \begin{pmatrix} \beta_1 \dots \beta_n \\ \bar{\beta}_1 \dots \bar{\beta}_n \end{pmatrix}. \tag{31}$$

Similar lemmas might be obtained for pfaffians and permanents, but they are not needed in electrodynamics, to which we mainly restrict this work.

3. – Evaluation of divergent integrals.

3·1. – We call *regularization* any procedure which yields finite results for an otherwise divergent integral, regardless of the physical or mathematical considerations which may have led to its adoption. We propose to find the nature of the additional requirements that a regularization procedure must satisfy, in order that it may act as a *renormalization*—a procedure, that is, for which suppressed divergent terms are proved to amount only to modifications of the parameters, masses and charges, in terms of which the original « unrenormalized » theory was written.

These requirements will restrict, of course, the variety both of possible procedures and of renormalizable theories; infinitely many different procedures are expected to exist, due to the always extant possibility of additional « finite renormalizations ». In the present work we discuss only the general conditions that a regularization must fulfil in order to satisfy necessary consistency requirements and to produce the typical behavior exhibited by formula (47); this formula will be the starting point, in the continuation to this work, for a detailed study of renormalizable theories, which will also lead to more specific conditions on the prescription itself. We shall prove there that the correct solution of the mathematical problem presented by the system of coupled hyperbolic differential equations for the kernels of field theory requires indeed the adoption of such a prescription.

As consistency requirements we list the following:

a) The regularization procedure must have a clear-cut mathematical definition; this must depend not upon physical peculiarities of the special processes which are being treated, but only upon general analytical properties of the integrands. It must be such as to make possible an investigation of the properties of the regularized integrals, say the study of majorants of them.

b) It must yield automatically, when applied to the perturbative expansions (11) (provided the other requirements to be stated in the sequel to this work are also fulfilled), the renormalized expansions; and eliminate thus the need for investigations on the nature of the single divergent contributions which it discards.

c) When the integrals in the branching equations (20)–(25) are performed in accordance with this prescription, the solutions (if any) of these equations (or of equations which approximate them) must be the renormalized kernels. Furthermore, the same results must obtain regardless of whether one searches for solutions of equations (20)–(25) as they stand, or of solutions of any other equation or set of equations obtained from them by any possible combination and iteration. Finally, the formal perturbative expansions calculated with this prescription must satisfy identically all such equations, where the integrals are regularized with the same rule.

d) All symmetry properties of integrals must be conserved.

In this way, once also the specific requirements to be studied later are fulfilled, so that the prescription is known indeed to be a renormalization, its adoption will eliminate consistently all divergencies; nor will it be necessary to study the problem of renormalization anew for any new approximation to equations (20)–(25) which one may wish to consider, as has been the case thus far. On the other hand,

e) we do *not* expect our prescription to yield *ipso facto* the experimental values for the parameters of the theory (masses and charges)—an additional finite renormalization may, in general, be necessary.

This is the price we pay to keep our branching equation *linear* throughout; it will be seen later that no serious inconvenient is caused thereby, while the advantages of adopting this point of view are tremendous. The whole question of renormalizability is changed into a much simpler one, as will be discussed later.

3·2. – Our investigation is based upon the use of configuration, rather than of momentum space. While it may still be convenient to resort to the latter for actual computations, use of the first presents great advantages in a general discussion, because all symmetries are thereby exhibited clearly [1] and the nature of possible divergencies is more easily identified and dealt with.

The divergencies we propose to remove are *only* the so-called ultraviolet divergencies, which originate whenever an integrand (say, any term of (11)) becomes too singular upon confluence of two or more of its points for the integral to converge (by *confluence of order h* [7] we denote a situation in which $h+1$ points either coincide or fall on the light cone of one another). Other infinities may appear because the overall space-time volume of integration is infinite: such is the case of vacuum fluctuations and of infrared divergencies, which we ignore in this discussion by tacitly assuming that some device is used to dispose of them, or to remove at least their consideration to a later stage.

The major difficulties we have to meet lie in the combinatorics; the situation would be nearly hopeless if we were to consider all possible sorts of *disjoint* confluences (*i.e.* in different points) which arise when studying, say, an expansion (11). We can, however, take advantage of equation (23) to bypass this obstacle, in a way which is best shown by discussing a simple mathematical model of the actual theory, which retains only those difficulties of the theory which stem from the combinatorics. This is done in the next section; the discussion will also display our method of attacking the problem, which consists in studying jointly the perturbative expansions and the branching equations, in order to achieve a greater clarity in the identification of the suppressed terms with the standard renormalizative corrections. Finally, this will indicate in which way an actual integration procedure must be formulated so as to meet the consistency requirements listed in the preceding section.

3·3. *A combinatorial model.* – This model was already discussed briefly in Ref. ([7]), Section 3·2. Its consideration here will serve to elucidate the behaviour of multiple integrals under regularization, out of which all other results follow quite naturally.

An essential property of the perturbative expansions (11) is that the N-th order integrand is a *symmetric function* of its arguments $\xi_1, \xi_2, \ldots, \xi_N$; this legitimates the free exchange of orders of integration, which is highly questionable for single Feynman graphs and gives rise indeed then to many ambiguities. This feature is of paramount interest for us, and shall be kept in the model.

We obtain our model by replacing throughout in the perturbative expansions and in all the branching equations each variable with a numerical index, which is denoted, for convenience, with the same symbol as the corresponding variable if this is external, or with $i, j, \ldots$ if it is an integration variable; integrations become, accordingly, summations over the *same* range for all indices. We omit, for simplicity, the γ matrices. Confluence reduces to mere coincidence of indices.

(11) becomes thus:

$$k_{N_0 P_0}(\lambda) = \sum_N \frac{\lambda^N}{N!} \sum_{i_1 i_2 \ldots i_N} f^{N_0 P_0}_{i_1 i_2 \ldots i_N}, \tag{32}$$

where

$$f^{N_0 P_0}_{i_1 i_2 \ldots i_N} \tag{33}$$

is symmetrical in all indices and

$$f^{N_0 P_0}_{i_1 \ldots i_N} = 0 \qquad \text{if } N \neq N(P_0) \tag{34}$$

Eq. (23) becomes

$$\frac{\partial}{\partial\lambda} k\begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0} = \sum_i k\begin{pmatrix} x_1 \dots x_{N_0} i \\ y_1 \dots y_{N_0} i \end{pmatrix} i\, t_1 \dots t_{P_0} . \tag{35}$$

It is very easy to see, then, that

$$\sum_{i_1 \dots i_N} f^{N_0 P_0}_{i_1 \dots i_N} = \sum_{h=1}^{N} \binom{N-1}{h-1} \sum_i \sum_{i'_1 \dots i'_{N-h} \neq i} f^{P_0 N_0}_{\underbrace{i \dots i}_{h}\, i'_1 \dots i'_{N-h}} , \tag{36}$$

this being simply another way of counting terms.

Formula (36) is the keystone of all our treatment of renormalization; symmetry is essential for its validity. (36) *leaves unchanged* all confluences among indices i', but isolates and classes all confluences according to the order in $i \neq i'_1 \dots i'_{N-h}$. The next important step is the equally trivial remark that:

$$\sum_i \sum_{(i') \neq i} = \sum_{(i')} \sum_i' \qquad \text{(inversion of sums)}, \tag{37}$$

where $(i') \equiv i'_1 \dots i'_{N-h}$, and the summations are performed in *reversed order*, with $\sum_i'$ indicating that the summation over i is now restricted the those values of i which are not taken already by some i': the indices i' now are *not* restricted in anyway, $\sum_i'$ is thus a summation performed with a well defined prescription, which determines unambigously, for any configuration $i'_1, \dots, i'_{N-h}$, the permissible values for i. This is, in its simplest form, the analogue of a « partie-finie integration rule » (see later); it is important to evidence explicitly the fact that the meaning of $\sum_i'$ is different for different values of $N - h$, and that the number of terms it gives rise to varies also with the configuration $i'_1, \dots, i'_{N-h}$.

Substitution of (36) into (35) gives:

$$\frac{\partial}{\partial\lambda} k\begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} t_1 \dots t_{P_0} = \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \sum_i k'\begin{pmatrix} x_1 \dots x_{N_0}\ \underbrace{i \dots i}_{h+1} \\ y_1 \dots y_{N_0} \end{pmatrix} \underbrace{i \dots i}_{h+1}\, t_1 \dots t_{P_0} , \tag{38}$$

where k' denotes that now the kernels are evaluated from their perturbative expansions with the rule: *for any given i, sum only over values of indices $\neq i$.*

(38) is to be compared with (35): we discover the remarkable fact, that adoption of this rule ($k \to k'$) classes all terms isolated away by it in increasing order of confluence, and that these group together again into *kernels*,

the formal properties of which are entirely known to us. This finding is the only reason why this line of approach proves successful.

Divergencies are far from being removed at this stage: the kernels k' still contain all confluences among indices i'. Our work with actual field theories will consist first in finding integration prescriptions which are analogous to (36), then in proving that « renormalizability » simply means that all terms at r.h.s. of (38) can be re-arranged so that they give rise to contributions that either vanish, or can be incorporated into additions to the « unrenormalized » parameters and into multiplicative factors for the kernels. A change of functions and parameters—which we simulate here by writing bars over all quantities—will change therefore, in a renormalizable theory, (38) into

$$\frac{\partial}{\partial\bar{\lambda}}\,\bar{k}\left(\begin{matrix}x_1 \dots x_{N_0}\\ y_1 \dots y_{N_0}\end{matrix}\middle|\, t_1 \dots t_{P_0}\right) = \sum_i \bar{k}'\left(\begin{matrix}x_1 \dots x_{N_0}\ i\\ y_1 \dots y_{N_0}\ i\end{matrix}\middle|\, i\, t_1 \dots t_{P_0}\right). \tag{39}$$

(In our model, (39) may just mean that we impose that the theory is modified so that all terms at r.h.s. of (38) vanish for $\hbar > 0$).

At this stage, we can make use of (37). The kernels k' are, thus far, defined only through their perturbative expansions: this amounts therefore to agree that the operation to be performed first, when the r.h.s. of (39) is substituted with its expansion, is the sum over i, now written $\sum_i'$. We find thus:

$$\frac{\partial}{\partial\bar{\lambda}}\,\bar{k}\left(\begin{matrix}x_1 \dots x_{N_0}\\ y_1 \dots y_{N_0}\end{matrix}\middle|\, t_1 \dots t_{P_0}\right) = \sum_i' \bar{k}\left(\begin{matrix}x_1 \dots x_{N_0}\ i\\ y_1 \dots y_{N_0}\ i\end{matrix}\middle|\, i\, t_1 \dots t_{P_0}\right), \tag{40}$$

in which $\sum'$, as defined by (37), depends upon the configuration of all other indices and acquires the rôle of a partie-finie rule.

We have thus that renormalization (the use of this word at this place will be justified in the sequel) amounts simply to replacing $\sum$ with $\sum'$: (35) becomes (40); this leads, of course, to different solutions (and different values for the parameters, upon comparison of theory with experiment), as is denoted by the bars.

The last step is to consider all the *branching equations* (20)–(25) written with the new summation symbol $\sum'$. Consider first equations (40): they can, like equations (35), be considered, upon iteration, as the generators of all perturbative expansions; it is evident then that, in our model, this gives rise to series

$$\bar{k}_{N_0P_0}(\lambda) = \sum_N \frac{\lambda^N}{N!} \sum_{i_1}' \sum_{i_2}' \dots \sum_{i_N}' f^{N_0P_0}_{i_1 i_2 \dots i_N} = \sum_N \frac{\lambda^N}{N!} \sum_{i_1 \neq i_2 \neq \dots \neq i_N} f^{N_0P_0}_{i_1 i_2 \dots i_N}. \tag{41}$$

The perturbative expansions written with $\sum'$ in the place of $\sum$ are therefore free from « divergent » (confluent) terms, and satisfy identically the whole set

of branching equations, if these are written also with $\sum'$. This proves that the same procedure that regularizes (or renormalizes) the perturbative expansions acts exactly in the same way on the branching equations, and conversely: that is, that all results are actually independent of perturbative arguments, although it is expedient to start from these in order to get a clearer understanding of the question.

We gather also, from this discussion, that all that is required to achieve this result is that (36), or some formula having equivalent combinatorial properties, holds. (36) is important also in another respect: it proves that it is then indifferent, in a given N-fold integral, to perform one integration at a time and drop terms, or to drop terms from the N-fold integral as a whole. In the last version we find here, in a nutshell, the analogue of Salam's prescriptions for the subtraction of overlapping divergencies, the mathematical origin of which could not be seen in momentum space but is trivially recognized here as due to the fact that higher confluences in configuration space, in an N-fold integral, range from order 1 to order $N-1$.

It should be evident, at this point, that any regularization procedure which correlates divergent integrals to regularized integrals through a relation similar to (36)—and, of course, is such as to secure the equivalence of (35) with (40), as was supposed in the discussion of this model—satisfies *ipso facto* also the requirements *b*), *c*) *d*) of Sect. 3·1. We conclude that this relation is the mathematical formulation of the conditions expressed verbally by these requirements. This solves our first problem, and gives an immediate way of testing whether a procedure that satisfies requirement *a*) (which it must do by definition, to be an acceptable candidate) and the set of conditions specific to a given theory (such as will be stated in the sequel to this work) fulfils also all our general consistency requirements.

4. – General conditions on regularization.

4·1. – The work of Section 3 indicates that the general requirements which are necessary (but not sufficient) for a regularization to become a renormalization can be satisfied as a matter of sheer combinatorics, provided the prescription acts on integrals so as to permit a separation of finite from divergent parts similar to that which was straightforward in the model. We must now give rules to this effect; the previous considerations have shown how to disentangle algebra from analysis. After this, it will remain to find in the sequel to this work, which additional conditions are necessary for formula (47), of which (38) is the combinatorial model, to take the form (49), of which (40) is the model: this will lead to the last set of conditions, which restrict both theories and prescriptions alike.

Since renormalization is expected to be possible, if at all, in many different ways—an additional finite renormalization being contemplated if necessary—it is convenient to discuss it first with a formalism of general validity, leaving undetermined as far as possible the specification of the rule which is actually used.

We start, again, from an analysis of perturbative expansions, and aim at a definition of regularization such that it may reproduce, when operating on multiple integrals, the same properties which were trivially valid in the model; the same arguments used there suffice to prove that all our general requirements are then satisfied, in particular independence from perturbative techniques. As a check, we shall show, in the sequel to this work, that any prescription such as is envisaged here gives indeed the correct formulation of the problem presented by the differential branching equations.

The N-th order term of the perturbative expansion of kernel has the form

$$\int d\xi_1 \ldots \int d\xi_N F^{(N)}(\xi_1, \xi_2 \ldots, \xi_N; x_1 \ldots, t_{P_0}) , \tag{42}$$

where the integrand is a *symmetric* function of $\xi_1, \xi_2, \ldots, \xi_N$, and takes a special form in each theory: *e.g.*, (11) for electrodynamics. Symmetry in the integration variables is the most important property: our rules will, and need, be formulated only in the case that integrands are symmetric; this secures also exchangeability of integrations, and permits to consider products of distributions.

The « finite part » of multiple integrals must be defined so that identical results obtain whether finite parts are taken for a variable at a time, or whether the N-fold integral is treated as a whole (as is the case with Salam's procedure). Consider then first confluences of $\xi_2, \xi_3, \ldots, \xi_N$ with ξ_1, disregarding confluences (and the corresponding divergencies) among the $\xi_2, \xi_3, \ldots, \xi_N$ themselves (cf. the model); write, at first formally:

$$\int d\xi_k F^{(N)}(\xi_1, \ldots, \xi_k, \ldots, \xi_N; x_1, \ldots, t_{P_0}) = \int_1 d\xi_k F^{(N)}(\xi_1, \ldots, \xi_k, \ldots; x_1, \ldots, t_{P_0}) + \tag{43}$$

$$+ D_1^k F^{(N)}(\xi_1, \ldots, \xi_k, \ldots, \xi_N; x_1 \ldots, t_{P_0}) ,$$

or, for short:

$$\int d\xi_k = \int_1 d\xi_k + D_1^k , \tag{44}$$

where it is understood in (44) that all operators act upon symmetric functions of $\xi_1, \ldots, \xi_N$.

The only well defined part of (43) of (44) is $\int_1 d\xi_k$: this is the symbol we choose to denote that a rule is adopted for its evaluation which eliminates *all, and only*, the divergent parts, if any, which arise because ξ_k is confluent with ξ_1. D_1^k is then the corresponding divergent contribution. This point has been already discussed in the Introduction; we remark here that the present notation is intended to cover all conceivable prescriptions, including cases where it is *not* required that $D_1^k F \equiv 0$ if there are actually no divergencies upon confluence. (Note that ξ_k is a *dummy* variable for both $\int_1 d\xi_k$ and D_1^k).

All we need in order that the same combinatorics as in the model be valid is thus that

$$(45) \quad \int d\xi_1 \dots \int d\xi_N F^{(N)}(\xi_1, \dots, \xi_N; \dots) =$$

$$= \int d\xi_1 \left[\int_1 d\xi_2 + D_1^2 \right] \dots \left[\int_1 d\xi_N + D_1^N \right] F^{(N)}(\xi_1, \dots, \xi_N; \dots) =$$

$$= \sum_{h=1}^{N} \binom{N-1}{h-1} \int d\xi_1 D_1^2 \dots D_1^h \int_1 d\xi_{h+1} \dots \int_1 d\xi_N F^{(N)}(\xi_1, \dots, \xi_N; \dots) .$$

(45) is the counterpart of (35), and holds provided

$$(46) \quad \int_1 d\xi_h D_1^k = D_1^h \int_1 d\xi_k \left(= D_1^k \int_1 d\xi_h \text{ because } \xi_h \text{ and } \xi_k \text{ are dummies} \right),$$

for *symmetric* integrands.

The requirements *b*), *c*), *d*), are therefore fulfilled provided (46) is satisfied by the regularization procedure. We have only, thus, to choose this so that it satisfies requirement *a*) and (46) (and the conditions which will be stated in a following report), to be sure that it is actually a renormalization.

In the next section we discuss some classes of prescriptions which fulfil (46). For the combinatorial part which is under discussion here, all results obtained in the study of the model can be taken over, with the evident necessary changes. We find, for the case of electrodynamics:

$$(47) \quad \frac{\partial}{\partial \lambda} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \, t_1 \dots t_{P_0} \right) = \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \int d\xi \sum \gamma^{\xi} \gamma^1 \dots \gamma^{\iota} D_{\xi}^1 \dots D_{\xi}^h \cdot$$

$$\cdot K^{(\xi)} \left(\begin{matrix} x_1 \dots x_{N_0} \xi\, \xi_1 \dots \xi_h \\ y_1 \dots y_{N_0} \xi\, \xi_1 \dots \xi_h \end{matrix} \middle| \, \xi \xi_1 \dots \xi_h t_1 \dots t_{P_0} \right),$$

which is the counterpart of (38); thus far, the kernels $K^{(\xi)}$ are defined through their formal perturbative expansions (11), in which all integrals on the new

dummy variables ξ' thus introduced are replaced by $\int\limits_{\xi} d\xi'$. All divergencies other than those arising from confluences in ξ are still contained in the r.h.s. terms; the latter are classed in increasing order of confluence in ξ.

The actual investigation of renormalizability, as was hinted to when discussing the model, will be based upon the remark that

$$\int d\xi_1 \int\limits_1 d\xi_2 \dots \int\limits_1 d\xi_l = \int d\xi_2 \dots \int d\xi_l \int d\xi_1 \tag{48}$$

into which (37) is now changed: $\sum'$ becomes $\int d\xi$, which denotes the *finite part calculated from ξ-integration* and depends upon the configuration of $\xi_1, \xi_2, \dots, \xi_N$ (and upon l). (48) is justified by the same arguments which lead to (37) and by the remark that, in the calculation of a finite part, it is irrelevant whether $\xi_h \to \xi$, or whether $\xi \to \xi_h$.

A regularization can be a renormalization, if both the regularizing prescription and the theory are such that, after appropriate changes of functions and parameters, (47) can be put into the equivalent form

$$\frac{\partial}{\partial \bar{\lambda}} \bar{K}\left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0}\right) = \int d\xi \sum \gamma^{\xi} \bar{K}\left(\begin{matrix} x_1 \dots x_{N_0} \ \xi \\ y_1 \dots y_{N_0} \ \xi \end{matrix} \middle| \xi \, t_1 \dots t_{P_0}\right). \tag{49}$$

After this is shown to be the case, the considerations made on the model apply without changes. The passage from (47) to (49), when it is possible, is a major combinatorial task and will be treated in the continuation to this work.

4˙2. – Classes of prescriptions for regularization which fulfil requirements *a*) and (46) are discussed, in general terms, in Ref. ([7]), Sections **3**˙2. We describe here another such class, which includes (and generalizes to multiple integrals) the concept of *partie finie* as given first by HADAMARD and is the counterpart, in configuration space, of the subtractive techniques which are standard in momentum space.

We remark first that any relation such as (46), which equates quantities which are actually infinite, has only a formal value: it can acquire a precise meaning if all infinite quantities are considered as the limits of finite expressions, for all determinations of which (and thus also in the limits) the equality holds. This occurs naturally if a proper use is made of the definition of singular integrals, or if, equivalently, convenient parameters or convergence factors are temporarily introduced into the integrands.

For the sake of simplicity, and without loss of generality, we discuss only the case of a function of some external variables (not explicitly indicated)

and of three internal variables. We suppose then that $\int_1 d\xi_h$ is defined by $(\eta_{hk} = \xi_h - \xi_k)$:

$$\int_1 d\xi_3 F(\xi_1, \xi_2, \xi_3) = \int d\xi_3 (F(\xi_1, \xi_2, \xi_3) - F_{\eta_{31}}(\xi_1, \xi_2, \xi_3)) , \tag{50}$$

where $F_{\eta_{31}}$ is a suitable subtractive term, devised so to remove *only* the divergence which occurs where $\eta_{31}^2 \to 0$, and not necessarily the others. A parametrization, or some such device, will keep (50) finite as long as is necessary, as was just discussed. Clearly:

$$D_1^3 F(\xi_1, \xi_2, \xi_3) = \int d\xi_3 F_{\eta_{31}}(\xi_1, \xi_2, \xi_3) . \tag{51}$$

Likewise:

$$\int_1 d\xi_2 \int_1 d\xi_3 F(\xi_1, \xi_2, \xi_3) = \int d\xi_2 \left[\int_1 d\xi_3 F - \left(\int_1 d\xi_3 F \right)_{\eta_{21}} \right] , \tag{52}$$

where now:

$$D_1^2 \int_1 d\xi_3 F(\xi_1, \xi_2, \xi_3)) = \int d\xi_2 \left(\int_1 d\xi_3 F \right)_{\eta_{21}} . \tag{53}$$

(53) is to be compared with:

$$\int_1 d\xi_2 D_1^3 F(\xi_1, \xi_2, \xi_3) = \int d\xi_2 [D_1^3 F - (D_1^3 F)_{\eta_{21}}] . \tag{54}$$

(53) equals (54) if:

$$\int d\xi_2 \int d\xi_3 F_{\eta_{31}} = \int d\xi_2 \left(\int d\xi_3 F \right)_{\eta_{31}} . \tag{55}$$

We suppose, further, that the subtracted term $F_{\eta_{31}}$ is uniquely determined by the singularity of F in $\eta_{31}^2 = 0$, and that it is given by the first sum at r.h.s. of:

$$F(\xi_1, \xi_2, \xi_3) = \sum_r d_r(\eta_{31}) f_r(\xi_1, \xi_2) + \sum_r c_r(\eta_{31}) g_r(\xi_1, \xi_2) ; \tag{56}$$

then also, because of the symmetry of F:

$$F(\xi_1, \xi_2, \xi_3) = \sum_r d_r(\eta_{21}) f_r(\xi_1, \xi_3) + \sum_r c_r(\eta_{21}) g_r(\xi_1, \xi_3) , \tag{57}$$

and $F_{\eta_{21}}$ obtains from $F_{\eta_{31}}$ upon permutation of ξ_2 and ξ_3.

Substitution of (57) at r.h.s. of (55) gives, when one looks for the part which diverges at η_{21}^2, the same expression which is obtained if $F_{\eta_{31}}$ is taken from (56) and introduced at l.h.s. of (55): *q.e.d.*

A prescription which satisfies requirement *a*), (50) and (56) is readily obtained if (56) is meant to denote the Laurent expansion of $F(\xi_1, \xi_2, \xi_3)$ in powers of the four dimensional variable η_{31}. Some care must be had, of course, in handling $F(\xi_1, \xi_2, \xi_3)$, which, as a distribution, is the boundary value of an analytic function of its variables when parameters $i\varepsilon^2$ (which express causality) vanish; but this offers no serious difficulty.

In practice, it will be convenient to apply (48), with variables $\xi_2, \xi_3, \ldots, \xi_N$ in non-confluent positions. After $\int d\xi_1$ is performed, the rule can be applied again to the resulting symmetric function of $\xi_2, \xi_3, \ldots, \xi_N$; then again, to the resulting function of $\xi_3, \xi_4, \ldots, \xi_N$, etc., until after $N-1$ iterations of the whole procedure, regularization is achieved, It will be proved in the next work that this is also a renormalization.

4·3. – We give, in conclusion and as an elucidation of our interpretation of renormalization, a brief discussion of two examples which exhibit, in a simple manner, the same behavior that we shall find typical of all renormalizable theories.

a) *Fermi field in external E.M. field.* The first traces of the iterated fermion free propagator are infinite ([8]). The standard device of attributing them the value zero (or any other finite value) can be readily interpreted as a $\int d\xi$ prescription. The thing is utterly trivial, and we need not digress on it here; it is relevant, though, to remark that by so doing we recognize that the problem of solving a hyperbolic equation does require the introduction of such a prescription.

b) *Hilbert integral equation.* This example is of interest because it shows, in a very simple way, how changing an ordinary Riemann integral—which is, under the circumstance, meaningless, just as in the formal branching equations of field theory——into a P.V. integral gives a well defined meaning to the equation and introduces « wave function » and « charge » renormalization.

Consider the equation:

$$\varphi(x) + \lambda \int \frac{\varphi(\xi)}{x - \xi} \, d\xi = f(x) \,, \tag{58}$$

where the integral ranges over, say, a finite interval. (58) is meaningless as it stands: all traces of $1/(x-\xi)$ are divergent.

We note, at this point, that our $\int d\xi$ can be interpreted, if it is conveniently particularized, as a generalization of the concept of P.V. integral (this point

will be discussed to a greater extent in the future). In this case, we have only a logarithmic divergence, and can thus identify the two concepts:

$$\int \mathrm{d}\xi \equiv \text{P.V.} \int \mathrm{d}\xi \,.$$

It is known that an equation like (58), if written with P.V. integrals, is correctly stated and has well-defined solutions. Such is:

$$\overline{\varphi}(x) + \overline{\lambda} \int \frac{\overline{\varphi}(\xi)}{x - \xi} \,\mathrm{d}\xi = f(x) \,. \tag{59}$$

The question is, whether and how one can compensate for the « mistake » originally made by writing (58) with an ordinary integral, so as to change it into the form (59). We have, that is, to find the relation between $\int$ and $\int$. This is very simple here: just remind that

$$\text{P. V.} \int \frac{\varphi(\xi)}{x - \xi} \,\mathrm{d}\xi = \lim \left(\int_{-\infty}^{x - \varepsilon'} + \int_{x + \varepsilon''}^{+\infty} \right) \frac{\varphi(\xi)}{x - \xi} \,\mathrm{d}\xi \,,$$

with the restriction $\varepsilon' = \varepsilon'' = \varepsilon$, when $\varepsilon \to 0$.
But the requirement $\varepsilon' = \varepsilon''$ is imposed only because one wishes to destroy the term

$$\lg \left| \frac{\varepsilon'}{\varepsilon''} \right| \cdot \varphi(x) \,.$$

The same result can be obtained if that term is suppressed, rather than with the additional requirement $\varepsilon' = \varepsilon''$, by simply subtracting it, when of course ε', ε'' are left arbitrary but dealt with consistently; that is, by writing:

$$\int \frac{\varphi(\xi)}{x - \xi} \,\mathrm{d}\xi = \int \mathrm{d}\xi \, \frac{\varphi(\xi)}{x - \xi} \,\mathrm{d}\xi + d \cdot \varphi(x) \,, \tag{60}$$

where we put, for short, $\lg |\varepsilon''/\varepsilon'| = d$ and leave it understood that $\int$ is calculated for the same (arbitrarily vanishing) ε', ε'' that came into the definition of d. The « divergent »—or, better, as we see from this example, « ambiguous »—part is given by

$$D^{\xi} \frac{\varphi(\xi)}{x - \xi} = d \cdot \varphi(x) \,.$$

Substitution of (60) into (58) gives:

$$(1+\lambda d)\varphi(x)+\lambda\int\frac{\varphi(\xi)}{x-\xi}\,d\xi=f(x)\,, \tag{61}$$

which reduces to (59) if one takes:

$$\varphi(x)=\frac{\overline{\varphi}(x)}{1+\lambda d}\,, \tag{62}$$

$$\overline{\lambda}=\frac{\lambda}{1+\lambda d}\,. \tag{63}$$

(62) is a « wave function renormalization » and (63) a « charge renormalization », in the familiar language of field theory. This example shows—and we shall prove in the next work that this is the case for all renormalizable theories—that if the equation is written in the « correct » form (59) to begin with, there is no need of (62) and (63), which serve only to compensate the faulty use of ordinary integrals in (58). A few more remarks may be in order. We note that $\overline{\lambda}$ in (59) is unknown *a priori*—experiment will decide on its numerical value. Thus, we may decide to use, instead of (60), the « subtractive » prescription (finite integration domain):

$$\int'\frac{\varphi(\xi)}{x-\xi}\,d\xi=\int d\xi\,\frac{\varphi(\xi)-\varphi(x)}{x-\xi}\,;\quad D\,\frac{\varphi(\xi)}{x-\xi}=\varphi(x)\int\frac{d\xi}{x-\xi}=\delta\cdot\varphi(x); \tag{64}$$

we obtain a result which differs from (60) by a *finite* contribution $(\delta-d)$ to the « divergent part ». Use of $\int'$ instead of $\int$ leads to an equation similar to (59), which reduces to (59) through a *finite* renormalization of type (62), (63). But if the parameter $\overline{\lambda}$ is to be deduced from experiment, this will make no difference: no matter what rule is chosen to evaluate $\int$, all numerical predictions of the theory remain the same, provided $\overline{\lambda}$ is given the appropriate value each time.

It is also quite instructive to repeat the same considerations on the formal Neumann or Fredholm expansions of the solutions of (58) and (59). This we leave to the reader, as a much more general analysis will be presented very soon.

* * *

Thanks are due to Dr. G. Fano for discussions on this subject, notably on the last example mentioned.

RIASSUNTO

Si formulano metodi generali che permettono di studiare le condizioni sotto le quali una regolarizzazione diviene una rinormalizzazione. Ciò accade ogni qualvolta il procedimento verifica condizioni di due tipi: un primo tipo, di natura assai generale, che deriva solo da requisiti combinatorici ed analitici; un secondo, che circoscrive insieme teorie e possibili procedimenti. Questo lavoro dà una trattazione esauriente delle condizioni del primo tipo e prepara lo studio di quelle del secondo, che verrà presto pubblicato. La conclusione principale sarà la dimostrazione che il problema delle divergenze ultraviolette viene completamente superato, riducendosi esso a quello di una appropriata ridefinizione del concetto di integrale, quale viene richiesta dalla teoria delle equazioni iperboliche. La trattazione è matematicamente rigorosa ed affatto indipendente da metodi perturbativi; essa pone la teoria su di una salda base per susseguenti investigazioni quantitative.

Further Investigation of a High Energy Jet.

G. Bozoki, G. Domokos, E. Fenyves, A. Frenkel and E. Gombosi

Central Research Institute of Physics - Budapest

D. Bebel, K. Lanius and H. W. Meier

Deutsche Akademie der Wissenschaften zu Berlin
Kernphysikalisches Institut - Zeuthen

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We have recently reported [1] on a high energy jet of the type $0+16\alpha$ found in the « I » Stack. As a supplement to this work the energies of secondary particles were determined by relative and single scattering measurements as well as by the analysis of the secondary electron-photon cascades.

Relative scattering measurements were carried out on 7 tracks of the narrow cone. Using the maximum possible cell-length of 4000 μm the relative scattering value does not exceed the noise level and thus only the lower limits of the energies could be determined (Table I).

Single scattering measurements could be carried out on 3 tracks of the diffuse cone. The scattering values obtained for two of the tracks were, however, inside the limits of error equal to the value of the spurious scattering. This latter value was estimated by measuring — besides the relative scattering — the single scattering of the tracks of the narrow cone ($D^2_{ss} = D^2_{sing} - D^2_{r.l}$) (*).

In the case of track no. 14 the influence of the distortion of the emulsion could not be eliminated and thus the energy value determined must be also taken as a lower limit. In column 4 of Table I there are listed the lower limits of transversal momenta ($p_\perp$) of secondary particles obtained from the measured angles and the estimated energies in the scattering measurements. The energy and transversal momentum values of secondary particles as obtained by scat-

(*) It is worth-while to mention that the value of spurious scattering for cell lengths greater than 1000 μm was inside the limits of error equal the value obtained by Brisbout *et al.* [2] in the « I » Stack. The dependence of spurious scattering values on the cell length (t) has the form of a power law: at^n ($n = 1.2 \pm 0.3$) for cell lengths 500 μm $\leqslant t \leqslant$ 3000 μm.

(1) G. Bozoki, G. Domokos, E. Fenyves, E. Gombosi, K. Lanius and H. W. Meier: *Intern. Conf. on Mesons and Recently discovered Particles* (Padua-Venice, 1957), p. XIII-20.

(2) H. A. Brisbout, C. Dahanayake, A. Engler, P. H. Fowler and P. B. Jones: *Nuovo Cimento*, **3**, 1400 (1956).

TABLE I.

No.	θ_{Lab} (rad)	Scattering measurements			Secondary interactions		
		E_{Iab} (GeV)		$p_\perp$ (GeV/c)	Type of interact.	E_{Lab} (GeV)	$p_\perp$ (GeV/c)
1	$0.6 \cdot 10^{-3}$	> 100	relative scattering	> 0.1			
2	0.8	> 100		> 0.1			
3	2.2	> 100		> 0.2			
4	2.3	> 100		> 0.2	1+8p	1300 ± 900	3.0 ± 2.0
5	2.8	> 100		> 0.3			
6	5.3	> 80		> 0.4			
7	6.1	> 80		> 0.5			
8	8.5	—		—	4+5p	60 ± 50	0.5 ± 0.4
9	10	> 8	single scattering	> 0.1			
10	30	> 8		> 0.2			
11	40	—		—			
12	110	—		—			
13	160	—		—			
14	170	> 1.3 single scattering		> 0.2	5+?p		
15	460	—		—			
16	730	—		—			
	$2.3 \cdot 10^{-3}$				13+14n	400 ± 200	1.6 ± 0.8

tering measurements are not in disagreement with the energy and transversal momentum values estimated for some of them from their secondary interactions (see column 6 and 7 of Table I and ref. [1]). One secondary interaction produced by an uncharged particle (13+14n) is also listed in Table I.

The energy transferred into neutral π-mesons was roughly estimated in ref. [1] by measuring the energies of electron pairs in the narrow cone and found to be of the order of magnitude of 10^{12} eV. A more detailed analysis of the soft cascades developed at a depth of 2.4 cascade units from the origin of the jet, according to the method used by Pinkau [3] showed that the value of the total energy transferred into the π^0-component is between $6 \cdot 10^{11}$ eV and $6 \cdot 10^{12}$ eV, the most probable value being about $2.5 \cdot 10^{12}$ eV.

The question of the primary energy has been reinvestigated using the two centre models [4,5]. In Table II the results of the calculations of γ_c are given together with the data obtained by the method of Castagnoli *et al.* [6] in first approximation and by the maximum likelihood method of Dilworth *et al.* [7].

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[3] K. PINKAU: *Nuovo Cimento*, **3**, 1285 (1956).

[4] P. CIOK, T. COGHEN, J. GIERULA, R. HOLYNSKI, A. JURAK, M. MIĘSOWICZ, T. SANIEWSKA and J. PERNEGR: *Nuovo Cimento*, **10**, 741 (1958).

[5] J. BURMEISTER, K. LANIUS and H. W. MEIER: *Nuovo Cimento*, **11**, 12 (1959).

[6] C. CASTAGNOLI, G. CORTINI, C. FRANZINETTI, A. MANFREDINI and D. MORENO: *Nuovo Cimento*, **10**, 1539 (1953).

[7] C. C. DILWORTH, S. L. GOLDSACK, T. F. HOANG and L. SCARSI: *Nuovo Cimento*, **10**, 1261 (1953).

Consequently, assuming a nucleon-nucleon-collision we get for the primary energy values between $2.7 \cdot 10^{12}$ and $1.3 \cdot 10^{13}$ eV. As pointed out in [1] the angular distribution in the C.M. system is strongly anisotropic and does not deviate significantly from the distributions expected on the basis of Landau and Heisenberg's theories. In consequence of the small number of shower particles, however, a two-peak distribution corresponding to the two-centre models can not be excluded.

Taking for the primary energy $1.3 \cdot 10^{13}$ eV the highest value given in Table II ($\gamma_c = 80$) and for the lower limit of the sum of the secondary energies $1.2 \cdot 10^{12}$ eV, we obtain — the big statistical error apart — for the lower limit of the inelasticity coefficient 0.1. It seems, however, to be more probable, that the inelasticity coefficient is higher than this value and equals or is even bigger than 0.5, this last value having also a very great statistical error.

The above considerations show that in high energy jets — except those very few cases, where the energies of all secondaries can be measured with a relatively great accuracy [8,9] — in general only the order of magnitude of primary and secondary energies, transversal momenta and inelasticity coefficients can be determined.

A significant increase of the experimental material might be hoped from stacks the dimensions of which allow to follow secondaries through several mean free path lengths and make it possible in this way to measure the energies of nearly all secondary particles.

TABLE II.

	CASTAGNOLI *et al.* [6]	Maximum likelihood method [7]	Fireball-model		Model of exc. nucl.	
			n_1/n_2 (*)	[4]	n_1/n_2 (*)	[5]
γ_c	59	80	7/9	75	7/9	40
			8/8	59	8/8	39
			9/7	46	9/7	37

(*) n_1, n_2: assumed number of particles in the inner and diffuse cone.

[8] R. G. GLASSER, D. H. HASKIN and M. SCHEIN: *Phys. Rev.*, **99**, 1555 (1955).

[9] A. DEBENEDETTI, C. M. GARELLI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, **4**, 1124 (1956).

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On the Anomalous Behavior of d(γ, p)n near 15 MeV.

S. H. Hsieh (*) and C. R. Lin (**)

(*) *Physical Institute, Nagoya University - Nagoya*
(**) *Department of Phyics, Tokyo University - Tokyo*

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Recent experiment on the photo-disintegration of the deuteron shows that the ratio a/b which represents the angular distribution,

$$\mathrm{d}\sigma/\mathrm{d}\Omega = (a + b \sin^2\vartheta)(1 + 2\beta\cos\vartheta)$$

has a maximum at $E_\gamma = 15$ MeV and a minimum at $E_\gamma = 17$ MeV (1). Here we illustrate its reason from physical point of view without touching to the detailed numerical computation (which is now in progress).

Speaking in a few words, the anomalous behavior of a/b at such energies is due to that the overlaping integral $\int W_d r U_F \mathrm{d}r$ (*) of the transition ${}^3D_1 \to {}^3F_2$ has a maximum at $E_\gamma \simeq 15$ MeV.

Now why does it cause the anomalous behavior of a/b? It may be understood easily by the following arguments.

In order to help the understanding let us quote the well known maximum of σ_d at $E_\gamma - \varepsilon \simeq 2$ MeV. At low energies the integrals which contribute to σ_{ed} are only

$$\int (U_d + X_{li} W_d) r U_{li} \mathrm{d}r = \int f_{di} r U_{li} \mathrm{d}r$$

which represents the overlap of $f_{di} r$ with U_{li} (2,3). Just near the threshold energy U_{li} is situated at very large distances, and the overlap is very small. But the overlapping integral increases rapidly with energy, and has maximum at $E_\gamma - \varepsilon \simeq 2$ MeV. As the overlapping integral decreases rapidly above $E_\gamma - \varepsilon \simeq 2$ MeV, the maximum of σ_{ed} is very remarkable.

Similar situation also occurs for the integral $\int W_d r U_F \mathrm{d}r$. This overlapping integral has a maximum at $E_\gamma \simeq 15$ MeV. As this integral contributes much to the isotropic part at $E_\gamma > 10$ MeV (4), we find a maximum of a/b at $E_\gamma \simeq 15$ MeV. As the contribution of this integral to σ_F is negligible at $E_\gamma \simeq 15$ MeV, σ_F has no anomaly at $E_\gamma \simeq 15$ MeV.

The non-isotropic part decreases very

(1) A. Whetstone and J. Halpern: *Phys. Rev.*, **109**, 2072 (1958).

(*) The symbols used here are same to those used in the reference (2).

(2) S. Hsieh: *Prog. Theor. Phys.*, **18**, 637 (1957).

(3) For the formulas of σ_{ed} see N. Austern: *Phys. Rev.*, **108**, 973 (1958).

(4) J. Swart and J. Marshak: *Phys. Rev.*, **111**, 272 (1958); S. Hsieh: to be published.

rapidly with energy so that a/b increases rapidly with energy ([5]), and as the overlapping integral $\int U_F r W_d \mathrm{d}r$ decreases very slowly at $E_\gamma > 15$ MeV, the maximum and the minimum of a/b at $E_\gamma \simeq 15$ MeV and 17 MeV are not very notable.

([5]) L. ALLEN: *Phys. Rev.*, **98**, 705 (1955).

Note added in proof.

K. FORD and J. WILLS have pointed out that the use of a point-nucleus mesic atom wave function for $\varphi_\mu(r)$ is not justified. The use of their wave function obtained by numerical integration, reduces the capture rates calculated in Table I, by 6%. We wish to thank them for this pre-publication communication.

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Pauli-Gürsey Transformations and Rotations in Charge Space.

E. Corinaldesi and L. A. Radicati

Istituto di Fisica dell'Università - Pisa
Istituto Nazionale di Fisica Nucleare - Sezione di Pisa

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It has been shown by Heisenberg and Pauli [1] that the equation

$$\gamma_\nu \partial_\nu \psi \pm l^2 \gamma_\mu \gamma_5 \psi (\bar{\psi} \gamma_\mu \gamma_5 \psi) = 0 , \tag{1}$$

($\bar{\psi} = \psi^+ \gamma_4$) is invariant under Pauli-Gürsey transformations [2]

$$\psi' = a\psi + b\gamma_5 \psi^+ , \tag{2}$$

with

$$|a|^2 + |b|^2 = 1 . \tag{3}$$

The present note deals with the isomorphism of (2) to the three-dimensional rotation group.

Instead of (1) it is convenient to study the equation

$$\gamma_\nu \partial_\nu \Phi \pm \frac{l^2}{4} \{(\gamma_\mu \gamma_5 \Phi) , \quad (\bar{\Phi} \gamma_\mu \gamma_5 \Phi)\} = 0 , \tag{4}$$

where

$$\Phi = \begin{pmatrix} \psi \\ \gamma_5 \psi^+ \end{pmatrix} , \qquad \bar{\Phi} = \Phi^+ \gamma_4 .$$

Equation (4) is equivalent to (1) and its Hermitian conjugate with a suitable symmetrization of the non-linear term.

The transformation (2) takes the form

$$\Phi' = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \Phi = (\alpha I + i \boldsymbol{\rho}^* \cdot \boldsymbol{\beta}) \Phi , \tag{5}$$

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[1] W. Heisenberg and W. Pauli: preprint (1958); see also: H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder and K. Yamazaki: preprint (1959).

[2] We use the Majorana representation of the γ's.

where I, ϱ_1, ϱ_2, ϱ_3 are the usual 2×2 Pauli matrices and

$$\alpha = \operatorname{Re} a\,, \qquad \beta_1 = \operatorname{Im} b\,, \qquad \beta_2 = -\operatorname{Re} b\,, \qquad \beta_3 = \operatorname{Im} a\,;$$

ϱ_i^* is the complex conjugate of ϱ_i. The unitarity condition (3) is equivalent to

$$\alpha^2 + \boldsymbol{\beta}^2 = 1\,.$$

For the special class of infinitesimal Pauli-Gürsey transformations

$$\Phi' = (I + i\boldsymbol{\rho}^* \cdot \boldsymbol{\beta})\Phi\,, \tag{6}$$

the unitary operator S which generates (5)

$$\Phi' = S\Phi S^{-1}\,, \tag{7}$$

has the form

$$S = I + 2i\boldsymbol{\beta} \cdot \boldsymbol{J}\,. \tag{8}$$

From (6), (7) and (8) follows

$$[\boldsymbol{J}, \Phi] = \tfrac{1}{2}\boldsymbol{\rho}^*\Phi\,. \tag{9}$$

It can easily be shown that the components of $\boldsymbol{J}$ obey the commutation relations of an angular momentum. For instance, using (9) twice, one gets

$$([J_2, J_3], \Phi) = \tfrac{1}{2}(-\varrho_2^*[J_3, \Phi] + \varrho_3^*[J_2, \Phi]) = i[J_1, \Phi]\,.$$

This shows that

$$[J_2, J_3] - iJ_1\,,$$

commutes with (5), and must therefore be a multiple of the unit operator which without loss of generality can be taken to be zero.

An explicit expression for $\boldsymbol{J}$ can be obtained in the usual way by equating to zero the variation

$$\delta\int\frac{1}{4}\left[\bar{\Phi}\gamma_\nu\partial_\nu\Phi \pm \frac{l^2}{2}(\bar{\Phi}\gamma_\mu\gamma_5\Phi)^2\right]\mathrm{d}^4x = -\frac{i}{2}\int\boldsymbol{\beta}\cdot\partial_\nu(\bar{\Phi}\gamma_\nu\boldsymbol{\rho}^*\Phi)\,\mathrm{d}^4x\,,$$

of the Lagrangian under local transformations (8) with $\boldsymbol{\beta}$ vanishing on the boundary. This gives the continuity equation

$$\partial_\nu j_\nu = 0\,, \tag{10}$$

with

$$\boldsymbol{j}_\nu = \frac{1}{4i}\,\bar{\Phi}\gamma_\nu\boldsymbol{\rho}^*\Phi\,.$$

It can be verified that (10) is a consequence of the equation of motion (1). From (10) it follows that

$$-i\int \boldsymbol{j}_4 \mathrm{d}^3x\,,$$

is a constant of the motion and can be identified with $\boldsymbol{J}$.

In terms of ψ its components read:

$$J_1 = -\frac{1}{4}\int [\psi^+\gamma_5\psi^+ + \psi\gamma_5\psi]\,\mathrm{d}^3x\,,$$

$$J_2 = -\frac{i}{4}\int [\psi^+\gamma_5\psi^+ - \psi\gamma_5\psi]\,\mathrm{d}^3x\,,$$

$$J_3 = -\frac{1}{4}\int [\psi^+\psi - \psi\psi^+]\,\mathrm{d}^3x\,.$$

Since J_1, J_2, J_3 are constants of the motion, one can use the anticommutation relations

$$\{\psi_\alpha(\boldsymbol{x}, t), \psi^+_\beta(\boldsymbol{x}', t)\} = \delta_{\alpha\beta}\delta(\boldsymbol{x}' - \boldsymbol{x})\,,$$

to show that

$$[J_i, J_k] = i\varepsilon_{ikl}J_l\,.$$

Under the transformation

$$C\psi C^{-1} = \psi^+$$

(« charge conjugation »),

$$J_1 \to J_1\,, \qquad J_2 \to -J_2\,, \qquad J_3 \to -J_3\,.$$

This shows that C amounts to a rotation by 180° around the first axis in charge space.

The theory is not invariant under arbitrary local Pauli-Gürsey transformations. The situation is similar to that discussed by YANG and MILLS [3] and, more recently, by SALAM and WARD [4]. Invariance can be achieved by introducing a vector field B_μ whose interaction with the ψ-field is obtained by replacing ∂_μ by $\partial_\mu - i\varepsilon B_\mu$ in the original equation (1).

This field would have the same properties as that of SALAM and WARD. Its introduction is, however, contrary to the spirit of Heisenberg's approach.

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[3] C. N. YANG and R. L. MILLS: *Phys. Rev.*, **96**, 191 (1954).

[4] A. SALAM and J. C. WARD: *Nuovo Cimento*, **11**, 568 (1959).

IL NUOVO CIMENTO Vol. XIII, N. 3 1° Agosto 1959

On Gravitational Radiation (*).

A. Peres

Department of Physics, Israel Institute of Technology - Haifa

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It was recently shown [1] that Einstein's equations have approximate solutions that represent a system of particles interacting with their own gravitational radiation. However, at each stage of the approximation procedure, one has to solve a Poisson equation, and there is a considerable freedom of choice of solutions, each representing a possible motion and a gravitational field belonging thereto [2]. Only one of these solutions behaves at infinity as purely outgoing waves [3]; the remaining ones also contain incoming waves. However, it is difficult to determine which solution is the correct one because the n-th term of a series expansion into powers of (v/c) behaves in the wave zone as R^{n-2}, and no boundary conditions for each stage of the procedure are known. The purpose of the present note is to give a criterion which partly removes this ambiguity.

A solution behaving at infinity like $f(t-R/c)/R$ has an expansion $f(t)/R - f'(t)/c + \ldots$. It is therefore reasonable to stipulate that if a term such as $f(t)/R$ occurs at some approximation stage, one adds $-f'(t)/c$ at the next stage. This method is not sufficient in general, but it gives unambiguous results up to the seventh order. As a consequence, one has to modify the fields that were previously used and one finds that the radiative correction to the acceleration is also altered. The radiated energy is now positive and agrees with results obtained by other methods [4]. The fact that one has previously obtained a negative radiated energy should be ascribed to the presence of incoming gravitational waves, which were absorbed by the particles.

This, and related topics, will be dealt with in detail in a paper to be published in this journal.

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(1) A. Peres: *Nuovo Cimento*, **11**, 617, 644 (1959).

(2) A. E. Scheidegger: *Phys. Rev.*, **99**, 1883 (1955).

(3) V. A. Fock: *Teoria Proctranstva, Vremeni i Tyagotenia* (Moscow, 1955), p. 441.

(4) L. Landau and E. Lifshitz: *The Classical Theory of Fields* (Cambridge Mass., 1951), p. 331.

LIBRI RICEVUTI E RECENSIONI

J. G. DELCROIX – *Introduction à la théorie des gas ionisés*, Ed. Dunod, Paris.

Sebbene siano ormai decenni che la fisica dei gas ionizzati costituisce un fertile campo di studio per ricercatori sia teorici che sperimentali, forse mai come ora essa ha attratto l'attenzione dei fisici.

Per rendersi conto della ricchezza di questo campo basti pensare che, come un tempo ha dato origine allo studio della spettroscopia ed alla astrofisica, così oggi il gas ionizzato (plasma) è considerato quello stato della materia (il quarto, dopo quelli solido, liquido e gassoso) sul quale si fondano le più rosee speranze dell'umanità sempre bisognosa di energia, in quanto in essa si spera che possa avvenire, così come avviene nelle stelle, la produzione, nel nostro caso controllata, di energia per fusione di nuclei leggeri.

Il fatto che si tratti di uno stato della materia differente da quello gassoso è abbastanza chiaro se si pensa che, per la presenza delle cariche elettriche, in un plasma le forze predominanti sono di origine elettromagnetica. Dal punto di vista pratico ciò è forse un vantaggio, perchè la maggior parte degli effetti studiati dalla meccanica quantistica (tranne l'interazione con la radiazione) possono essere ignorati e lo stesso urto tra particelle cariche è regolato dalla semplice legge di Coulomb. Dal punto di vista sia della previsione che dell'intuizione dei fenomeni che avvengono nei gas ionizzati le cose non sono però molto semplici perchè le forze elettromagnetiche che agiscono e si sviluppano nel plasma impartiscono alle particelle cariche dei moti che non hanno un parallelo nel caso dei gas neutri.

La descrizione teorica delle proprietà dei gas ionizzati assume due aspetti complementari: mentre da un lato il comportamento macroscopico si può far derivare da una dettagliata analisi microscopica delle interazioni tra le particelle costituenti il gas ionizzato, dall'altra si può costruire una sorta di idrodinamica del plasma stesso pensato come un fluido sottoposto a forze di origine elettromagnetica.

Va da se che i due punti di vista debbono avere una zona « di saldatura » e che questa debba essere costituita — analogamente al caso della teoria dei gas neutri — dalla statistica, per esempio sotto forma di equazione di Boltzmann.

A seconda della natura dei fenomeni che si vogliono studiare ciascuno dei metodi presenta precipui vantaggi.

Il libretto di J. L. DELCROIX introduce alla teoria dei gas ionizzati partendo dal punto di vista dell'analisi microscopica dei problemi.

Esso parte infatti dalla classica descrizione dell'urto elastico tra due particelle ed evitando digressioni e senza insistere se non sull'essenziale conduce con tutta chiarezza e rigore attraverso lo studio delle funzioni di distribuzione, dei fenomeni collettivi, delle oscillazioni del plasma fino alla formulazione delle equazioni macroscopiche del plasma

stesso, cioè al punto di vista della meccanica del gas ionizzato pensato come un fluido.

E qui si ferma, pago di aver fatto da guida al lettore — senza avergli richiesto più di una normale conoscenza della fisica matematica — attraverso tutta una successione di fenomeni nuovi a cui il disordine particellare dà luogo quando è controllato dalle forze elettromagnetiche.

Il lettore che volesse poi vedere come questi vari fenomeni si coordinino ed interferiscano è rimandato dallo stesso autore del bel libretto ora visto all'altro volumetto pubblicato ora da Dunond: *Physique des gas complètement ionisés*, dello SPITZER. Questa opera è ormai tanto nota, sia agli specialisti in plasma che agli altri fisici, che non sarebbe forse il caso di parlarne se non in contrapposizione a quella del DELCROIX. Questi due libri, infatti, mentre hanno in comune una esposizione chiara, concisa, lineare, riguardo al contenuto sono invece complementari, in quanto il lavoro di SPITZER studia i gas ionizzati principalmente dal punto di vista della dinamica dei fluidi, considera cioè un fluido particolare formato da cariche elettriche dei due segni e sottoposto a forza di natura elettromagnetica. Questa meccanica nella quale per esempio l'equazione del moto del fluido determina la densità di corrente (elettrica) mentre invece la velocità del fluido stesso è determinata da una sorta di legge di Ohm generalizzata, costituirebbe null'altro che un groviglio di equazioni (non lineari) ed un quasi sterile sforzo per la mente del lettore se l'autore, con mano sapiente, non facesse constatare ad ogni passo come i complessi fenomeni che si possono studiare altro non sono che l'effetto del bilanciamento e della cooperazione di tanti fenomeni elementari (che per esempio nel libro di DELCROIX sono studiati estesamente).

Concludendo, bene ha fatto l'editore Dunod a presentare al pubblico francese questi due trattati, ciascuno di per sè pregevole, presi insieme per ora insostituibili.

U. ASCOLI BARTOLI

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